TECHNOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 62 NUMBER 4

CONTENTS

Mean Value Theorems and Linear Operators . . . . . PHILIP HARTMAN AND AUREL WINTNER 217
Criteria for Irrationality of Certain Classes of Numbers. II . . . . . P. H. DIANANDA AND A. OPPENHEIM 222
Some Inequalities Arising from a Generalized Mean Value Theorem . . . . . M. P. DRAZIN 226
Postulates and Mathematics . . . . . C. H. DENBOW 233
The Algebra of Semi-magic Squares . . . . . L. M. WEINER 237
On the Euler Characteristic of a Lie Algebra . . S. I. GOLDBERG 239
Mathematical Notes . . . . . D. G. DUNCAN, L. CARLITZ, JOHN JONES, JR., HANS SCHNEIDER 241
Classroom Notes . . G. M. PETERSEN, G. B. HUFF, I. A. BARNETT, ... M. J. PASCUAL, J. P. BALLANTINE, R. LARIVIERE 249
Elementary Problems and Solutions 254
Advanced Problems and Solutions 258
Recent Publications—College Text Books 265
News and Notices 288
The Mathematical Association of America 296
Report of the Treasurer for the Year 1954 296
New Members 299
November Meeting of the Philadelphia Section 301

APRIL 1955

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

CARL B. ALLENDOERFER, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	EDWIN HEWITT	E. P. STARKE
HOWARD EVES	L. M. KELLY	G. B. THOMAS
F. A. FICKEN	IVAN NIVEN	E. P. VANCE
WALLACE GIVENS	EDITH R. SCHNECKENBURGER	H. S. ZUCKERMAN
	GEORGE SPRINGER	

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, W. L. DUREN, JR., Tulane University
Honorary President, W. D. CAIRNS, Oberlin College
First Vice-President, H. S. M. COXETER, University of Toronto
Second Vice-President, G. B. PRICE, University of Kansas
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo
Editor, C. B. ALLENDOERFER, University of Washington

Additional Members of the Board of Governors: I. A. BARNETT, M. A. BASOCO, H. W. BRINKMANN, B. H. BROWN, S. S. CAIRNS, H. H. DOWNING, R. M. FOSTER, PHILIP FRANKLIN, R. F. GRAESSER, E. R. HEINEMAN, R. C. HUFFER, RALPH HULL, C. A. HUTCHINSON, S. B. JACKSON, R. D. JAMES, L. W. JOHNSON, P. S. JONES, R. E. LANGER, F. A. LEWIS, SAUNDERS MACLANE, K. O. MAY, E. J. MCSHANE, R. J. MICHEL, E. B. MILLER, C. W. MUNSHOWER, W. H. MYERS, C. O. OAKLEY, E. N. OBERG, MORRIS OSTROFSKY, F. A. RICKEY, C. E. SPRINGER, C. W. TRIGG, A. W. TUCKER, M. A. ZORN

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash., during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

MEAN VALUE THEOREMS AND LINEAR OPERATORS

PHILIP HARTMAN and AUREL WINTNER, The Johns Hopkins University

The object of this note is to isolate and systematize a procedure occurring in some apparently unrelated problems, such as the proof of the mean value theorem of differential calculus and Pólya's generalization [8] of it, and Zaremba's treatment [11] of solutions of the Laplace equation. As immediate consequences (without the use of the existence or positivity of the Green kernel, in which regard *cf.* [2]), there will result the mean value theorems of Blaschke [1] for the Laplacian, as well as mean value theorems for the heat equation operator and for other differential and "generalized" differential operators. The procedure to be dealt with has already been subject to some systematization in a paper of Pólya [9].

Let D and E be fixed subsets of a topological space S of points x and let D be dense in S. Let $\mathfrak{M}$ be a fixed linear manifold of (real- or complex-valued) functions $u=u(x)$ which are defined and continuous on S, and let $\mathfrak{N}$ be a fixed linear submanifold of $\mathfrak{M}$. Finally, for every u in $\mathfrak{M}$, let $L(u)$ be a function (of the position x) on the subset D of S. The continuity of the function $L(u)$ with respect to x is not assumed, nor is there any assumption of continuity for the operator L with respect to u.

In the applications below, D will be either an open set or an open set and part of its boundary (in an n-dimensioned Euclidean space), S the closure of D and E either the interior of D or D itself, L a differential or "generalized" differential operator, $\mathfrak{M}$ the set of continuous functions u on S for which $L(u)$ is defined (not necessarily continuous) on D, finally $\mathfrak{N}$ the set of those functions $u\in\mathfrak{M}$ which satisfy a certain boundary condition, that is, functions u possessing certain derivatives which, together with u, vanish on a part of the boundary of D.

The main theorem to be proved is as follows:

(*) *Let the operator L satisfy the following conditions:*

(i) *$L(u)$ is linear on $\mathfrak{M}$, that is,*

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2); \tag{1}$$

(ii) *there exists a solution $u=U_0$ of the boundary value problem*

$$L(U_0) = 1 \text{ on } D, \qquad \text{where } U_0 \in \mathfrak{N}; \tag{2}$$

(iii) *if $u\in\mathfrak{N}$ and $L(u)\neq 0$ on D, then $u\neq 0$ on E.*

Further, let u be any function of $\mathfrak{M}$ for which there exists a solution U of the boundary value problem

$$L(U) = 0 \text{ on } D, \qquad \text{where } U - u \in \mathfrak{N}. \tag{3}$$

Then, corresponding to every $x\in E$, there exists at least one point $\theta=\theta(x)$ of D satisfying

$$u(x) = U(x) + U_0(x)L(u(\theta)). \tag{4}$$

It is understood that $L(u(\theta))$ means the value of the function $L(u)$ at the point θ of D.

The assumption (iii) is analogous to Rolle's theorem and the assertion (4) to the mean value theorem in differential calculus.

Proof. Since (iii) implies that the solution U_0 of (2) does not vanish on E, it follows that, corresponding to a given point x of E, there exists a number A satisfying

$$u(x) - U(x) - U_0(x)A = 0.$$

On the other hand, the function $u-U-U_0A$ is in $\mathfrak{N}$, by (2) and (3), and vanishes at the point x of E. Hence (iii) assures that $L(u-U-U_0A)$ vanishes at some point θ of D. But the linearity of L and $L(U)=0$, $L(U_0)=1$ imply that $L(u(\theta))-A=0$. Hence (4) follows from the last formula line, and so (*) is proved.

(A) If L satisfies conditions (i)–(iii) of (*), then the following uniqueness theorem holds:

(iii *bis*) *Any solution u of the boundary value problem*

$$L(u) = 0 \text{ on } D, \qquad \text{where } u \in \mathfrak{N}, \tag{5}$$

satisfies $u \equiv 0$ on E.

For suppose that $u=v$ is a solution of (5) and that $v(x_0)\neq 0$ at some $x=x_0\in E$. Then, by (2) and (5), $U_0+cv\in\mathfrak{N}$ and $L(U_0+cv)=1\neq 0$ on D for any constant c. Since c can be chosen so that $u=U_0+cv=0$ at $x=x_0$, there results a contradiction to (iii).

On the other hand, (i), (ii) and (iii *bis*) do not imply (i)–(iii). In order to see this, let $D=E$ be the 1-dimensional open interval $0<x<5\pi/2$; S the closure of D; $\mathfrak{M}$ the set of continuous functions u on S having a continuous second derivative on D; $\mathfrak{N}$ the set of functions $u\in\mathfrak{M}$ satisfying $u(0)=u(5\pi/2)=0$; finally, $L(u)=d^2u/dx^2+u$. Then the solution of (5) is $u\equiv 0$. But the (only) solution of (2) is $U_0=1-2^{1/2}\sin(x+\pi/4)$, which vanishes at the point $x=\frac{1}{2}\pi$ of $E(=D)$.

Although (iii *bis*) does not imply (iii), one has:

(B) If L satisfies (i), (ii) and the following *relaxed* form of the *weak* maximum principle (iii *tre*), then L satisfies (iii).

(iii *tre*) *Let S be compact, let D be connected and let $u\in\mathfrak{N}$ imply that u vanishes on the complement $S-E$ of E. Then, if $L(u)>0$ on D, the function u has no maximum on E.*

Whenever (iii *tre*) is used, it is understood that the functions $u\in\mathfrak{M}$ are assumed to be real-valued. In order to see that (iii *tre*) implies (iii), let a $u\in\mathfrak{N}$ satisfy $L(u)\neq 0$ on D. It can be supposed that $L(u)>0$, for otherwise u could be replaced by $-u$. If (iii) does not hold, so that $u(x_0)=0$ for some $x_0\in E$, then $u(x_0)$

is either a maximum value of u or u has a positive maximum at some other point of E. Since this contradicts (iii *tre*), it follows that (iii *tre*) implies (iii).

Several illustrations of (*) will now be given. The functions u will always be assumed to be real-valued.

(I) Taylor's formula is valid in the following form: Let D be the 1-dimensional open interval $0<x<h$. Let $\mathfrak{M}$ be the continuous functions u on $0\leqq x\leqq h$ which possess continuous derivatives $du/dx, \cdots, du^{m-1}/dx^{m-1}$ on the half-open interval $0\leqq x<h$ and an mth order (not necessarily continuous) derivative d^mu/dx^m on the open interval $0<x<h$. Then, if $u\in\mathfrak{M}$, there exists at least one number θ satisfying $0<\theta<h$ and

$$u(h) = \sum_{k=0}^{m-1} u^{(k)}(0)h^k/k! + u^{(m)}(\theta)h^m/m!.$$

This fact (I), proved in [4] directly, follows from (*) by the choices $L(u) = du^m/dx^m$, $E=(0<x\leqq h)$ and $\mathfrak{N}$ the set of functions $u\in\mathfrak{M}$ satisfying $u(0) = u^{(1)}(0) = \cdots = u^{(m-1)}(0) = 0$.

(II) Pólya's extension [8] of Rolle's theorem is the case where $L(u)$ is an ordinary differential operator of the mth order with continuous coefficients on the closed interval $E=D=(0\leqq x\leqq 1)$; $\mathfrak{M}$ is the set of functions u possessing a continuous mth derivative on D; B denotes a set of m (not necessarily distinct) points $a_1\leqq a_2\leqq \cdots \leqq a_m$ of $S=(0\leqq x\leqq 1)$; finally, $u\in\mathfrak{N}$ means that u vanishes in the kth order (at least) at $x=a$, if $x=a$ is a point of B counted with multiplicity k. In effect, Pólya shows that if L has property (iii *bis*) for every choice of the m points of B, then L has properties (ii) and (iii) for every choice of B; that is, if the only solution of $L(u)=0$ having m zeros on $0\leqq x\leqq 1$ is $u\equiv 0$, then L has properties (ii) and (iii) for every choice of the m points of B.

(III) Let $D(=E)$ be the interior of the circle $x^2+y^2<1$ and B its boundary $x^2+y^2=1$. Let $u\in\mathfrak{M}$ mean that u is continuous on $S=D+B$ and that

$$L(u) = \lim_{h\to 0} \{f(x-h, y-h) + f(x+h, y-h) + f(x+h, y+h) + f(x-h, y+h) - 4f(x, y)\}/h^2$$

exists at $(x, y)\in D$, and let $u\in\mathfrak{N}$ mean that $u\in\mathfrak{M}$ and $u=0$ on B. Clearly, L has property (i), as well as (iii *tre*), hence (iii). Finally, property (ii) follows from the fact that if $u(x, y)$ has the second order partial derivatives u_{xx}, u_{yy} at a point, then $L(u)$ exists at that point and

$$L(u) = \Delta u, \quad \text{where} \quad \Delta u = u_{xx} + u_{yy} \tag{6}$$

(*cf.* [11], p. 174); so that the solution $u=U_0$ of (2) exists and is

$$U = \tfrac{1}{4}(x^2 + y^2 - 1). \tag{7}$$

If u is continuous on B, there exists a solution U of (3); in fact, U is the function which is harmonic in D and assumes the values of u on the boundary.

It is now easy to conclude Zaremba's theorem [11], which states that if u is continuous on $S=D+B$ and if $L(u)=0$ on D, then u is harmonic on D. In fact, this is an obvious consequence of the statement (4), which indeed shows that $u \equiv U$. Similarly, if $L(u)=2\pi f$ exists (and is continuous), then u is, up to an additive harmonic function, the logarithmic potential with density f; *cf.* [10], p. 735.

If, in Zaremba's theorem, the function $u(x, y)$ is chosen to be independent of y, there results a theorem of Hölder ([6], pp. 183–185) which states that if

$$\{u(x+h) - 2u(x) + u(x-h)\}/h^2 \to 0, \qquad h \to 0,$$

for every point x of an interval, then u is a linear function.

(IV) It is clear that the considerations of (III) can be repeated if the unit circle $x^2+y^2<1$ is replaced by any Jordan domain D. Similarly, if $L(u)$ is defined by (6) for functions u continuous on $D+B$ and having second order partial derivatives u_{xx}, u_{yy} on D, then, corresponding to any $(x, y) \in D$, there exists an $(\alpha, \beta) \in D$ satisfying

$$u(x, y) = U(x, y) + U_0(x, y)\Delta u(\alpha, \beta), \tag{8}$$

where U is harmonic in D and satisfies $U=u$ on the boundary B of D, while $\Delta U_0=1$ on D and $U_0=0$ on B. Note that the continuity of u_{xx} or u_{yy} is not assumed, nor is the existence of u_{xy}. The mean value theorem (8) is given by Pólya [9] under the assumption that u has continuous second order partial derivatives on D.

If D is particularized again, this time to $D=(x^2+y^2<r^2)$, then $U_0=\frac{1}{4}(x^2+y^2-r^2)$; *cf.* (7). Let

$$\mu_r(u) = (2\pi)^{-1} \int_0^{2\pi} u(r\cos\phi, r\sin\phi)d\phi. \tag{9}$$

Then $U(0, 0)=\mu_r(U)=\mu_r(u)$. Hence (8) becomes, for $(x, y)=(0, 0)$,

$$\mu_r(u) - u(0, 0) = r^2\Delta u(\alpha, \beta)/4, \quad \text{where} \quad \alpha^2+\beta^2<r^2. \tag{10}$$

This corresponds to a mean value theorem of Blaschke [1] who, however, imposes more stringent conditions on u.

Let $0<s<r$ and let r be replaced by s in (10). Then $\alpha=\alpha(s)$, $\beta=\beta(s)$ and $\alpha^2(s)+\beta^2(s)<s^2$. It is clear from (10) that $\Delta u(\alpha(s)), \beta(s))$ is a continuous function of $s(>0)$. Thus, if (10) is multiplied by sds, integration over $0<s<r$ gives, for some (α, β),

$$\lambda_r(u) - u(0, 0) = r^2\Delta u(\alpha, \beta)/8, \qquad \alpha^2+\beta^2<r^2, \tag{11}$$

where

$$\lambda_r(u) = \iint_{x^2+y^2<r^2} u(x, y)dxdy/(\pi r^2) = 2\int_0^r \mu_s(u)sds/r^2. \tag{12}$$

The relation (11) corresponds to another mean value theorem of Blaschke [1].

Let D be the unit circle $x^2+y^2<1$ and B its boundary. Let $u\in M$ mean that u is continuous on $S=D+B$ and that either

$$L(u) = \lim_{s\to 0} 4\{\mu_s(u(x+x_0, y+y_0)) - u(x_0, y_0)\}/s^2$$

or

$$L(u) = \lim_{s\to 0} 8\{\lambda_s(u(x+x_0, y+y_0)) - u(x_0, y_0)\}/s^2$$

exists at every point (x_0, y_0) of D. Finally, let $u\in\mathfrak{N}$ mean that $u\in\mathfrak{M}$ and that $u=0$ on B. By the procedure of (III), it is clear that if $u\in\mathfrak{M}$ and if $L(u)=0$ on D, then u is harmonic on D; *cf.* [1].

The preceding theorems have obvious extensions to spaces of dimensions $n>2$. Furthermore, Δu (and its generalizations) can be replaced by other elliptic differential operators; for example, by $\Delta u+$const. u, where the constant is negative and D is a Jordan domain (with sufficiently smooth boundary) or by $\Delta u+$const. u, where the constant is positive but D is sufficiently small; *cf.* [7], pp. 217–219.

(V) Consider the case of a suitable parabolic operator $L(u)$. Let $D(=E)$ be the square $(0<x<1,\ 0<y\leqq 1)$ and let $S=D+B$, where B is the sum of the lower $(y=0,\ 0\leqq x\leqq 1)$ and lateral ($x=0$ and $x=1$, $0\leqq y\leqq 1$) boundaries of D. Let $u\in\mathfrak{M}$ mean that u is continuous on the closure $D+B$ of D and that u_x, u_y, u_{xx} exist on D, and define $L(u)$ by

$$L(u) = \partial u, \quad \text{where} \quad \partial u = u_{xx} - u_y. \tag{13}$$

Let $u\in\mathfrak{N}$ mean that $u\in\mathfrak{M}$ and that $u=0$ on B. Since (i) is obvious, as is the maximum principle (iii *tre*), and since (ii) is easily verified, (*) implies the mean value theorem

$$u(x, y) = U(x, y) + U_0(x, y)\partial u(\alpha, \beta),$$

where $(x, y)\in D$, $(\alpha, \beta)\in D$, U satisfies $\partial U=0$ on D and $u=U$ on B, while U_0 satisfies $\partial U_0=1$ on D and $U_0=0$ on B; *cf.* Pólya [9].

As in (III), if (12) is replaced by

$$L(u) = \lim_{h\to 0} \{u(x+h, y) + u(x-h, y) + u(x, y-h^2) - 3u(x, y)\}/h^2$$

(*cf.* [5], pp. 598–601), it can be shown that if u is continuous on $D+B$ and if $L(u)=0$ on D, then the function u must have derivatives of arbitrarily high order, and satisfy $\partial u=0$, on D. Correspondingly, the last definition of $L(u)$ can be replaced by

$$L(u) = \lim_{h\to 0} \{[u_x(x+h, y) - u_x(x, y)]/h - [u(x, y) - u(x, y-h^2)]/h^2\},$$

where it is assumed that u is continuous on $D+B$ and that u_x and $L(u)$ exist on D; *cf.* Gevrey [3], p. 363 and p. 369.

References

1. W. Blaschke, Mittelwertsatz der Potentialtheorie, Jber. Deutschen Math. Verein., vol. 27, 1918, pp. 157–160.

2. K. Bleuler, Über den Rolle'schen Satz für den Operator $\Delta u+\lambda u$ und die damit zusammenhängenden Eigenschaften der Green'schen Funktion, Zürich Doctoral Thesis, Geneva, 1942.

3. M. Gevrey, Sur les équations aux dérivées du type parabolique, J. de Mathématiques, ser. 6, vol. 9, 1913, pp. 305–471.

4. P. Hartman, Remark on Taylor's formula, Bull. Amer. Math. Soc., vol. 51, 1945, pp. 731–732.

5. ——— and A. Wintner, On the inverse of the parabolic differential operator $\partial^2/\partial x^2-\partial/\partial t$, Amer. J. Math., vol. 75, 1953, pp. 598–610.

6. O. Hölder, Zur Theorie der trigonometrischen Reihen, Math. Ann., vol. 24, 1884, pp. 181–216.

7. F. Pockels, Über die partielle Differentialgleichung $\Delta u+k^2u=0$ und deren Auftreten in der mathematischen Physik, Leipzig, 1891.

8. G. Pólya, On the mean-value theorem corresponding to a given linear homogeneous differential equation, Trans. Amer. Math. Soc., vol. 24, 1922, pp. 312–324.

9. ———, Quelques théorèmes analogues au théorème de Rolle, liés à certaines équations linéaires aux dérivées partielles, Comptes Rendus, vol. 199, 1934, p. 655.

10. A. Wintner, On the Hölder restrictions in the theory of partial differential equations, Amer. J. Math., vol. 72, 1950, pp. 731–738.

11. S. Zaremba, Sur une propriété générale des fonctions harmoniques, Conférences de la Réunion Internationale des Mathématiciens, (1937), Paris, 1939, pp. 171–176.

CRITERIA FOR IRRATIONALITY OF CERTAIN CLASSES OF NUMBERS. II

P. H. DIANANDA and A. OPPENHEIM, University of Malaya, Singapore

Introduction. In a paper with the same title published recently in this MONTHLY,* criteria were obtained for the irrationality of the number x given by the series (1) wherein the a_i are integers and the b_i are positive integers. In particular were considered the series (1) where a_i, b_i satisfy (2) and (3) below. These criteria depended on the limits of the sequence $c_i=a_i/b_i$. But the criteria there found did not cover the case when each of the limits of the sequence c_i is rational and not 0 and not 1. Theorem A below deals with this case.

The series (1) may be called Cantor's series since Cantor [1] showed that for given integers $b_i\geqq 2$, a real number x can be expanded uniquely (if irrational) in the form (1) with the a_i satisfying (3). (If x is rational two expansions are possible). Cantor showed that if the b_i satisfy the condition that the product $b_1b_2\cdots b_i$ is divisible by an arbitrary integer q for some sufficiently large i (and

* This MONTHLY, vol. 61, 1954, pp. 235–241.

therefore for all greater i), then x is irrational unless $a_i=0$ (all large i) or $a_i=b_i-1$ (all large i) in which cases x is rational. He also showed that when the b_i are ultimately periodic, x is rational if and only if the a_i are ultimately periodic.

In Theorem B below we give a criterion which includes both these criteria of Cantor.

1. We confine ourselves for simplicity to the series (1) subject to the conditions (2) and (3).

$$x = a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \cdots, \tag{1}$$

$$b_i \geqq 2, \qquad (i = 1, 2, 3, \cdots), \tag{2}$$

$$0 \leqq a_i \leqq b_i - 1, \qquad (i = 1, 2, \cdots). \tag{3}$$

Criteria for irrationality were obtained which depended on the limits of the sequence

$$c_i = a_i/b_i$$

when $i \to \infty$. It was proved that x is irrational in each of the following cases: for some subsequence (i_n)

$c_{i_n} \to \xi$ (irrational),

$c_{i_n} \to 1$ provided $a_i < b_i - 1$ infinitely often,

$c_{i_n} \to 0$ provided $b_{i_n} \to \infty$ and $a_i > 0$ infinitely often.

The theorem which follows covers the remaining case when *every* limit of the sequence (c_i) is a rational number h/k where $0<h<k$, $(h, k)=1$.

THEOREM A. *If every limit of $c_i=a_i/b_i$ is a rational number h/k where $0<h<k$, $(h, k)=1$, then x is irrational except possibly when*

$$a_i = [hb_i/k]$$

for all large i in the subsequence for which $c_i \to h/k$.

In the excepted case x may be rational or irrational.

Proof. By the hypothesis there exists a subsequence (i_n) such that $c_i \to h/k$ and either

$$\frac{a_i}{b_i} > \frac{h}{k} \qquad \text{infinitely often,}$$

or

$$\frac{a_i + 1}{b_i} \leqq \frac{h}{k} \qquad \text{infinitely often}$$

in this subsequence. Hence $b_{i_n} \to \infty$.

Recalling that, by Lemma 2,

$$\frac{a_i}{b_i} \leqq x_i < \frac{a_i + 1}{b_i},$$

(for $a_i < b_i - 1$ infinitely often) we see that, for this subsequence,

$$x_i \to h/k, \qquad x_i \neq h/k.$$

If x is a rational number with denominator q, then each x_i is a rational number with denominator q. But a rational number with bounded denominator cannot tend to a rational h/k except by equality. The contradiction shows that x is irrational.

The excepted case is illustrated by two examples.

(i) $$a_i = i,\ b_i = 2i + 1\ (i = 1, 2, \cdots)\colon\ a_i/b_i \to \tfrac{1}{2}.$$

Here $x = \frac{1}{2}$; $a_i = [\frac{1}{2}b_i]$.

(ii) By Theorem 4 the number

$$y = \sum_{i=1}^{\infty} 1/b_1 \cdots b_i, \qquad (b_i = 3i + 2),$$

is irrational. Take $a_i = i$. Then

$$3x = \sum_{1}^{\infty} \frac{3i}{5 \cdots (3i + 2)} = 1 - y$$

is irrational so that x is irrational. Here plainly

$$c_i = a_i/b_i \to 1/3, \qquad a_i = [b_i/3].$$

2. Condensation. From the series (1), a series of precisely the same form may be obtained by grouping terms together thus:

$$\frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \cdots + \frac{a_{i_1}}{b_1 \cdots b_{i_1}} = \frac{A_1}{B_1}$$

where $B_1 = b_1 b_2 \cdots b_{i_1}$, $0 \leqq A_1 \leqq B_1 - 1$, these inequalities following at once from (3). By this process of condensation as it may be called we are led to

$$x = X = A_0 + \frac{A_1}{B_1} + \frac{A_2}{B_1 B_2} + \frac{A_3}{B_1 B_2 B_3} + \cdots$$

where

$$A_0 = a_0,\ B_1 = b_1 \cdots b_{i_1},\ B_2 = b_{i_1+1} \cdots b_{i_1+i_2}, \cdots,\ B_i \geqq 2,\ 0 \leqq A_i \leqq B_i - 1,$$

a series of precisely the same form as (1).

THEOREM B. *A necessary and sufficient condition that x given by* (1), *subject to* (2) *and* (3), *shall be rational is this: coprime integers h, k, $0 \leqq h \leqq k$, an integer N and a condensation shall exist such that*

$$A_i = \frac{h}{k}(B_i - 1)$$

for all $i \geqq N$.

The sufficiency of the condition is obvious. That it is necessary may be seen as follows. If x is rational with denominator $q \geqq 1$, every x_i is rational with denominator $\leqq q$. Hence there exists a subsequence (j_n) such that

$$x_{j_n} = h/k, \qquad (0 \leqq h \leqq k,\ k \geqq 1,\ (h, k) = 1),$$

since $0 \leqq x_i \leqq 1$.

Use the condensation defined by

$$i_1 = j_1 - 1,\ i_1 + i_2 - 1 = j_2,\ i_1 + i_2 + i_3 - 1 = j_3, \cdots .$$

For this condensation

$$x_{j_1} = X_2, \qquad x_{j_2} = X_3, \cdots ,$$
$$B_r X_r = A_r + X_{r+1},$$

so that

$$A_r = X_r(B_r - 1) = \frac{h}{k}(B_r - 1), \qquad (r = 2, 3, \cdots),$$

which plainly implies that $k \mid B_r - 1$. Thus the condition is necessary.

The criterion given in Theorem B is interesting since it includes the two criteria given by Cantor: (i) when the b_i are such that, for every integer q, $q \mid b_1 b_2 \cdots b_n$ for all large n, (ii) when the b_i are periodic.

In the first case the criterion is this: x is rational if and only if $a_i = 0$ $(i > i_0)$ or if $a_i = b_i - 1$ (all $i > i_0$).

In the second case the criterion is this: x is rational if and only if the a_i are ultimately periodic.

It is unnecessary to give the details of the deduction of these criteria from Theorem B.

Reference

1. G. Cantor, Über die einfachen Zahlensysteme, *Zeit. für Math.*, vol. 14, 1869, pp. 121–128; *Coll. Papers*, Berlin, 1932, pp. 35–42.

SOME INEQUALITIES ARISING FROM A GENERALIZED MEAN VALUE THEOREM

M. P. DRAZIN, Trinity College, Cambridge

1. Introduction. Let $f(x)$ be a given real function, defined and continuous on the closed interval $c \leqq x \leqq c+h$ (where $h>0$); suppose also that $f(x)$ is differentiable in the open interval $c<x<c+h$. Then the Mean Value Theorem asserts the existence of a real number ξ satisfying

$$f(c+h) - f(c) = hf'(c+\xi), \qquad 0 < \xi < h.$$

In this note we shall first establish an easy extension of this for higher derivatives; such extensions are presumably familiar (in one form or another) to many mathematicians, but the writer has not been able to trace any general statement in the literature. Next, temporarily abandoning the continuous variable and turning to the discrete case, we obtain a combinatorial identity (to which similar remarks apply) involving repeated differences of an arbitrary sequence.

These two results have some immediate corollaries of a rather more qualitative type. In particular, we use them to establish a class of inequalities which do not seem to be known, and which do not yield easily to the usual methods.

All functions considered and constructed below are to be understood as real functions of the variable x (defined over appropriate intervals of the real axis); we shall not always trouble to repeat this at each stage.

Given any non-zero h, we shall write

$$\Delta_h f(x) = \{f(x+h) - f(x)\}/h;$$

we shall in fact be concerned exclusively with *positive* h, but there is no difficulty in extending our results to the general case.

2. The Generalized Mean Value Theorem. Our point of departure is the following Theorem:

THEOREM 1. *Let $f(x)$ be continuous for $a \leqq x \leqq b$, and suppose that $f^{(n)}(x)$ exists in the open interval $a<x<b$ (where $a<b$ and n is some given positive integer). Then, given any n positive numbers $h_1, \cdots, h_n$, there is a sequence of real (but not necessarily continuous) functions $\xi_k(x)$ $(k=1, \cdots, n)$, respectively defined on the interval $a \leqq x \leqq b-h_1- \cdots -h_k$, and such that*

$$\left.\begin{aligned} &(1) \quad 0 < \xi_k(x) < h_1 + \cdots + h_k \\ &(2) \quad \Delta_{h_k} \cdots \Delta_{h_1} f(x) = f^{(k)}(x + \xi_k(x)) \end{aligned}\right\} \qquad \begin{aligned} &(a \leqq x \leqq b - h_1 - \cdots - h_k; \\ &k = 1, \cdots, n). \end{aligned}$$

Proof. The result for $n=1$ is just the statement of the Mean Value Theorem (as applied to $f(y)$ in the intervals $x \leqq y \leqq x+h_1$). Working by induction on n, we suppose the result proved for $n-1$ (and for *all* permissible $f, a, b, h_1, \cdots, h_{n-1}$). It will then be enough to suppose $\xi_1(x), \cdots, \xi_{n-1}(x)$ already defined (over their

respective intervals) satisfying (1), (2), and to find, for each x in the interval

$$I: a \leqq x \leqq b - h_1 - \cdots - h_n,$$

a number $\xi_n(x)$ satisfying

$$0 < \xi_n(x) < h_1 + \cdots + h_n,$$
$$\Delta_{h_n} \cdots \Delta_{h_1} f(x) = f^{(n)}(x + \xi_n(x)).$$

For any $x \in I$ the Mean Value Theorem, applied to the function

$$g(y) = \Delta_{h_{n-1}} \cdots \Delta_{h_1} f(y)$$

in the interval $x \leqq y \leqq x + h_n$, assures us of the existence of a number $\eta(x)$ satisfying

$$0 < \eta(x) < h_n,$$
$$\Delta_{h_n} g(y)\big|_{y=z} = g'(y)\big|_{y=z+\eta(z)} = \Delta_{h_{n-1}} \cdots \Delta_{h_1} f'(y)\big|_{y=z+\eta(z)}.$$

Also, by our induction hypothesis, applied to $f'(y)$ in the closed interval $x+\eta(x) \leqq y \leqq x+\eta(x)+h_1+\cdots+h_{n-1}$ (which lies strictly interior to the open interval $a<y<b$), there is a number $\zeta = \zeta_{n-1}(x+\eta(x))$ satisfying

$$0 < \zeta < h_1 + \cdots + h_{n-1},$$
$$\Delta_{h_{n-1}} \cdots \Delta_{h_1} f'(y)\big|_{y=z+\eta(z)} = f^{(n)}(x) + \eta(x) + \zeta).$$

Hence

$$\Delta_{h_n} \cdots \Delta_{h_1} f(x) = \Delta_{h_n} g(y)\big|_{y=z} = f^{(n)}(x + \eta(x) + \zeta),$$

and the theorem follows at once on taking $\xi_n(x) = \eta(x) + \zeta$.

COROLLARY. *If, in Theorem* 1, *a particular derivative* $f^{(k)}(x)$ *has constant sign in the open interval* $a<x<b$, *then* $\Delta_{h_k} \cdots \Delta_{h_1} f(x)$ *has the same constant sign throughout its range of definition* (*i.e. for* $a \leqq x \leqq b - h_1 - \cdots - h_k$).

Unfortunately, even for $n=1$, there is no simple converse of this Corollary (deducing a property of $f^{(k)}(x)$ from a hypothesis about the sign of $\Delta_{h_k} \cdots \Delta_{h_1} f(x)$, where $h_1, \cdots, h_n$ are of course always regarded as fixed). For we have no guarantee that the points $z = x + \xi_k(x)$ can be chosen so as to exhaust the interval $a<y<b$. It is easy, for the case $n=k=1$, to give a precise statement about the various possibilities, as folllows:

THEOREM 2. *Let* $f(x)$ *be continuous for* $a \leqq x \leqq b$, *and differentiable for* $a<x<b$. *Given any* h *with* $0<h<b-a$, *let* $Z = Z(f, a, b, h)$ *denote the set of all points* z *satisfying*

$$x < z < x + h, \qquad f(x + h) - f(x) = hf'(z)$$

for some x *in the range* $a \leqq x \leqq b-h$. *Then* Z *must contain at least two distinct points* z; *also* Z *can contain exactly two points for suitably chosen* f, a, b, h.

Proof. Let $P_z = (x, f(x))$ denote the point of the graph of $f(x)$ with ordinate

x, and consider the chord P_aP_{a+h} joining the points with ordinates a, $a+h$. There are only two possibilities: either (i) this chord contains in its interior a third point (P_c, say) of the graph, or (ii) it does not.

In case (i), by applying the Mean Value Theorem to the intervals

$$a \leqq x \leqq c, \qquad c \leqq x \leqq a + h,$$

we see that there are points z_1, z_2 satisfying

$$(1) \qquad a < z_1 < c, \qquad\qquad c < z_2 < a + h,$$

$$(2) \qquad f(c) - f(a) = (c - a)f'(z_1), \qquad f(a + h) - f(c) = (a + h - c)f'(z_2).$$

But, since P_c lies on the segment P_aP_{a+h}, the gradients $\{f(c)-f(a)\}/(c-a)$, $\{f(a+h)-f(c)\}/(a+h-c)$ of the subsegments P_aP_c, P_cP_{a+h} are both equal to the gradient $\{f(a+h)-f(a)\}/h$ of the full segment. Thus we may rewrite (2) as

$$f(a + h) - f(a) = hf'(z_1) = hf'(z_2);$$

also, by (1), z_1, z_2 are distinct points of the open interval $a<x<a+h$. In other words, Z contains the two distinct points z_1, z_2.

Suppose next that (ii) holds, *i.e.*, (by the continuity of $f(x)$) that the graph of $f(x)$ lies strictly to one side of the open segment P_aP_{a+h} for $a<x<a+h$. We may suppose without loss that the graph lies *above* the chord. By the Mean Value Theorem, Z contains at least one point in the interior of every subinterval of $[a, b]$ having length h. Suppose, if possible, that Z contains only one point, say z_0. This would imply, for *all* x with $a \leqq x \leqq b-h$, that

$$x < z_0 < x + h, \qquad f(x + h) - f(x) = hf'(z_0),$$

i.e., that

$$(3) \qquad b - h < z_0 < a + h,$$

and

$$(4) \qquad f(x + h) - f(x) = hf'(z_0) \qquad (a \leqq x \leqq b - h).$$

This is clearly absurd if $[a, b]$ has length $2h$ or more. But, in any case, substituting successively $x=a$, $x=a+\epsilon$ in (4) and subtracting, we deduce, for $0<\epsilon \leqq \min(b-a-h, h)$, that

$$f(a + h + \epsilon) - f(a + h) = f(a + \epsilon) - f(a) > \epsilon\{f(a + h) - f(a)\}/h,$$

since the graph lies above the chord. On letting $\epsilon \to 0$, it follows that

$$f'(a + h) \geqq \{f(a + h) - f(a)\}/h.$$

But also, since the graph of $f(x)$ lies above P_aP_{a+h} in every sufficiently small left neighborhood of $x=a+h$, we have

$$f'(a + h) \leqq \{f(a + h) - f(a)\}/h.$$

Hence in fact

$$f'(a+h) = \{f(a+h) - f(a)\}/h,$$

and so, taking $x=a$ in (4), we see that $f'(z_0)=f'(a+h)$; thus (4) may be rewritten

$$f(x+h) - f(x) = hf'(a+h), \qquad (a \leqq x \leqq b-h).$$

In particular, for any δ with $0<\delta<\min(b-a-h, h)$,

$$f(a+h+\delta) - f(a+\delta) = hf'(a+h),$$

and so Z contains the point $a+h$; also, by (3), $a+h \neq z_0$, so we have the desired contradiction.

We have now shown (directly in case (i), and by a *reductio ad absurdum* in case (ii)) that Z must always contain at least two points. Finally, our argument makes it clear that Z need not contain more than two points; a simple concrete example is provided by the function $f(x)=\sin x$, with $a=0$, $b=7$, $h=2\pi$.

3. An Identity. Given any sequence of numbers a_t $(t=0, 1, 2, \cdots)$, we write

$$\Delta^0 a_t = a_t, \qquad \Delta^1 a_t = \Delta a_t = a_{t+1} - a_t,$$

and define inductively

$$\Delta^k a_t = \Delta(\Delta^{k-1} a_t) \qquad (k = 2, 3, \cdots; t = 0, 1, \cdots).$$

THEOREM 3. *For any given sequence a_t, any given y, and any non-negative integer n, we have the identities*

$$I_{n,k}\colon (-1)^k \sum_{r=0}^{n-k} \binom{n-k}{r} y^r \sum_{s=0}^{k} \binom{k}{s} (-1-y)^s \Delta^{k-s} a_{r+s} = \sum_{t=0}^{n} \binom{n}{t} y^t a_t, \qquad (k = 0, 1, \cdots, n).$$

Proof. The identities can of course be verified by comparing coefficients of the powers of y and expanding the terms $\Delta^{k-s} a_{r+s}$ by Newton's formula. However, we shall give a proof by induction (the case $n=0$ being trivial).

We note first that, for $k=1, \cdots, n$,

$$\begin{aligned}
&\sum_{s=0}^{k} \binom{k}{s} (-1-y)^s \Delta^{k-s} a_{r+s} \\
&= \sum_{s=0}^{k} \left\{ \binom{k-1}{s} + \binom{k-1}{s-1} \right\} (-1-y)^s \Delta^{k-s} a_{r+s} \\
&= \sum_{s=0}^{k-1} \binom{k-1}{s} (-1-y)^s \Delta^{k-s} a_{r+s} + \sum_{s=0}^{k-1} \binom{k-1}{s} (-1-y)^{s+1} \Delta^{k-s-1} a_{r+s+1} \\
&= \sum_{s=0}^{k-1} \binom{k-1}{s} (-1-y)^s \{ \Delta^{k-s} a_{r+s} - (1+y) \Delta^{k-s-1} a_{r+s+1} \},
\end{aligned}$$

while also

$$\Delta^{k-s}a_{r+s} - (1+y)\Delta^{k-s-1}a_{r+s+1} = \Delta^{k-1-s}\{\Delta a_{r+s} - (1+y)a_{r+s+1}\}$$
$$= -\Delta^{k-1-s}(a_{r+s} + ya_{r+s+1});$$

hence

$$(-1)^k \sum_{r=0}^{n+1-k} \binom{n+1-k}{r} y^r \sum_{s=0}^{k} \binom{k}{s} (-1-y)^s \Delta^{k-s}a_{r+s}$$
$$= -(-1)^k \sum_{r=0}^{n-(k-1)} \binom{n-(k-1)}{r} y^r$$
$$\cdot \sum_{s=0}^{k-1} \binom{k-1}{s} (-1-y)^s \Delta^{k-1-s}(a_{r+s} + ya_{r+s+1}).$$

Suppose now that $I_{n,k}$ $(k=0, 1, \cdots, n)$ hold for a given n (and for *all* sequences), and consider $I_{n+1,k}$ $(k=0, 1, \cdots, n+1)$. Of these, $I_{n+1,0}$ is trivial, while for $k \geqq 1$, by applying $I_{n,k-1}$ (for the sequence $a_t + ya_{t+1}$) to the expression on the right of the identity we have just obtained, we find

$$(-1)^k \sum_{r=0}^{(n+1)-k} \binom{(n+1)-k}{r} y^r \sum_{s=0}^{k} \binom{k}{s} (-1-y)^s \Delta^{k-s}a_{r+s}$$
$$= \sum_{t=0}^{n} \binom{n}{t} y^t (a_t + ya_{t+1}) = \sum_{t=0}^{n+1} \binom{n+1}{t} y^t a_t$$

by an argument similar to that used above; in other words, $I_{n+1,k}$ holds.

The result now follows generally by induction.*

We shall not require Theorem 3 in its full generality, but only in the special case $k=n$, which it will be convenient to state separately:

THEOREM 4. *For any sequence a_t, and any y,*

$$\sum_{t=0}^{n} \binom{n}{t} y^t a_t = (-1)^n \sum_{s=0}^{n} \binom{n}{s} (-1-y)^s \Delta^{n-s}a_s \qquad (n = 0, 1, \cdots).$$

We note, in passing, a simple consequence of this. Applying Theorem 4 twice in succession, first for a_t, y, and then for $\Delta^{n-t}a_t$, $-1-y$, we obtain

$$\sum_{t=0}^{n} \binom{n}{t} y^t a_t = (-1)^n \sum_{s=0}^{n} \binom{n}{s} (-1-y)^s \Delta^{n-s}a_s$$
$$= (-1)^n \cdot (-1)^n \sum_{r=0}^{n} \binom{n}{r} \{-1-(-1-y)\}^r \Delta^{n-r}(\Delta^{n-r}a_r)$$

* It is clear that the properties of the real number system are only partially relevant to the argument; in fact the result still holds for sequences from any additive group which admits y as an operator.

$$= \sum_{r=0}^{n} \binom{n}{r} y^r \Delta^{n-r}(\Delta^{n-r} a_r).$$

Since this holds for all y, comparison of coefficients of powers of y gives:

COROLLARY. *For any sequence* a_t,

$$\Delta^{n-t}(\Delta^{n-t} a_t) = a_t \qquad (0 \leqq t \leqq n).$$

This result is perhaps slightly surprising at first sight, but it becomes intuitively clear from a consideration of the formal difference table of the a_t; alternatively, a more pedestrian check can be made by a direct application of Newton's formula (using the fact that, for fixed u,

$$\Delta^{n-t} \binom{n-t}{u} = (-1)^{n-t} \text{ if } u = n - t,$$

and is otherwise zero).

4. Some Inequalities. We shall now briefly discuss some joint consequences of the results of the two previous sections. First of all, Theorem 4 leads at once to

THEOREM 5. (i) *If* $(-1)^{n-s}\Delta^{n-s} a_s \geqq 0$ $(s = 0, 1, \cdots, n)$, *with at least one inequality, then*

$$\sum_{t=0}^{n} \binom{n}{t} y^t a_t > 0 \qquad (y > -1).$$

(ii) *If* $\Delta^{n-s} a_s \geqq 0$ $(s = 0, 1, \cdots, n)$, *with at least one inequality, then*

$$(-1)^n \sum_{t=0}^{n} \binom{n}{t} y^t a_t > 0. \qquad (y < -1).^*$$

We shall concern ourselves only with results of the type (i); there are precisely analogous results of type (ii). We shall also now consider only the case in which the a_t arise as the values at integer points of a given (and appropriately differentiable) function $f(x)$; but, again, a result similar to Theorem 7 below holds good for arbitrary discrete sequences (and can be similarly proved directly, or deduced as a corollary).

By Theorems 1 (with $h_1 = \cdots = h_n = 1$) and 4, we have

THEOREM 6. *Let* $f(x)$ *be a given function continuous over the interval* $0 \leqq x \leqq n$, *and suppose also that* $f^{(n)}(x)$ *exists in the open interval. Then there are numbers* $x_0, x_1, \cdots, x_n$, *with* $s < x_s < n$ $(s = 0, 1, \cdots, n-1)$, $x_n = n$, *such that, for all* y,

* If we do not demand at least one inequality in (i) and (ii), then (even if we also allow $y = -1$) we of course have parallel limiting results stating that the sums are non-negative. Also, if we insist on having inequality in the hypothesis for $s = 0$, we can allow $y = -1$ in each case and still be sure of having strict inequality in the conclusions. Similar remarks apply to Theorem 7 below.

$$\sum_{t=0}^{n}\binom{n}{t}y^t f(t) = (-1)^n \sum_{s=0}^{n}\binom{n}{s}(-1-y)^s f^{(n-s)}(x_s).$$

In analogy with Theorem 5(i), if $(-1)^{n-s}f^{(n-s)}(x) \geqq 0$ $(s=0, 1, \cdots, n)$, then we can deduce that the sum on the left is non-negative for $y \geqq -1$. A little more generally, we have:

THEOREM 7. *Let $f_1(x), \cdots, f_q(x)$ be given continuous functions over the interval $0 \leqq x \leqq n$, each differentiable n times in the open interval and positive for $x=n$; suppose also that*

$$(-1)^k f_p^{(k)}(x) \geqq 0 \qquad (p = 1, \cdots, q;\ k = 1, \cdots, n;\ n-k < x < n).$$

Then

$$\sum_{t=0}^{n}\binom{n}{t}y^t \prod_{p=1}^{q} f_p(t) > 0 \qquad (y > -1).$$

Proof. On writing $f(x) = \prod_{p=1}^{q} f_p(x)$, then $f(x)$ satisfies the conditions of Theorem 6. Also $f(n) > 0$, so, by Theorem 6, it is enough to show that

$$(-1)^{n-s}f^{(n-s)}(x) \geqq 0 \qquad (s = 0, 1, \cdots, n-1;\ s < x < n),$$

or, equivalently,

$$(-1)^k f^{(k)}(x) \geqq 0 \qquad (k = 1, \cdots, n;\ n-k < x < n);$$

and this follows immediately from the identity

$$(-1)^k f^{(k)}(x) = \sum_{\substack{k_1+\cdots+k_q=k \\ k_1\geqq 0, \cdots, k_q \geqq 0}} \frac{k!}{k_1! \cdots k_q!}(-1)^{k_1} f_1^{(k_1)} \cdots (-1)^{k_q} f_q^{(k_q)}.$$

As an illustration of Theorem 7, one set of functions satisfying the conditions is

$$f_p(x) = (1+\alpha_p x)^{-\beta_p} \qquad (p = 1, \cdots, q),$$

where the α_p, β_p can be any non-negative numbers. The writer is indebted to Mr. E. M. L. Beale for bringing to his notice the inequality corresponding to this choice of the f_p (for $\beta_1 = \cdots = \beta_q = \frac{1}{2}$); Beale had obtained an *ad hoc* proof for this particular case, not easily capable of extension to cover anything appreciably more general, and it was the writer's attempt to produce a more satisfying proof which led to this note.

POSTULATES AND MATHEMATICS

C. H. DENBOW, Ohio University

The common denominator of the views held by most mathematicians, with regard to the nature and authenticity of mathematical findings, seems to be the conclusion that mathematics has nothing to do with meaning or with truth, but is concerned only with validity. This view is so far removed from the convictions of most students and most laymen that a very difficult communication problem arises—one which concerns those who wish to improve mathematical education and also those who regret any fragmentation of knowledge into exclusive and mutually incomprehensible specialties.

Teachers of secondary or undergraduate college courses face clients who, like laymen in general, believe that mathematics is the sure method of arriving at infallible and eternal truths. For example, it is generally believed that the arm-chair methods of euclidean geometry are powerful enough to demonstrate the necessary behavior of distance, of congruent figures, *etc.*, beyond the outermost of the galaxies. Though this view has been obsolete for at least a century, it is well to recall here that thinkers of this persuasion once established the intellectual patterns of western civilization. For more than two thousand years believers in the methods of Euclid formulated the "axioms" of other fields, and from them attempted to deduce absolute truths and final systems in natural science, political science, and in ethics. Mathematics was considered, in that period, as the brightest star in the constellation of deductive knowledge.

We are especially concerned here to note the divergent paths followed by science and by mathematics after this period of rampant rationalism ended; after a long list of discoveries, including that of non-euclidean geometries, gave evidence that deduction from axioms was an untenable method of achieving absolute knowledge in mathematics or in science. In science the concept of "absolute truth" has since been replaced by that of "probable truth." Moreover, rationalism has been relegated to the status of being one of the many tools of science, together with experimentation, hypothesizing, *etc.*

It is not surprising that mathematicians should choose a far different solution to the problem posed by the downfall of the "absolute truth" philosophy. They were mindful that it was the inner harmonies of their subject, rather than experimental data or physical problems, which led, for example, to the discovery of quaternions and to the theory of groups. The belief in the absolute truth of euclidean geometry seems to have been secondary, in their thinking, to their pride in the logical beauty and consistency of geometry; attributes which passed unscathed through Lobachevsky's onslaught. In the emergent philosophy of mathematics they "took the cash" of inner harmony and beauty and "let the credit go" of conquering reality. Guided by the concepts of the postulational approach, they developed new sources of mathematical power and discovered many hidden treasures—and at the same time their aims and motivations, even

more than their findings, became increasingly inexplicable to the general public who are their sponsors.

The mere existence of non-euclidean geometry and abstract number systems can be used, of course, to shatter the faith of our clients in the absolute truth view of mathematics. Unfortunately, however, extremely few of them are mathematically mature enough to appreciate the abstract postulational approach as a substitute view. This great divergence in outlook creates frustration in students and/or teachers, especially when the latter adopt an uncompromising "this or nothing" attitude. A young college instructor, thoroughly imbued with the postulational philosophy, may find, for example, that his freshmen do not understand his remarks on irrational numbers. His premises about the nature of mathematics then force him to draw one of three conclusions. Either he has failed to state his postulates with precision, or his development of, say, Cantor sequences lacked logical rigor, or else his students are simply incapable of learning the topic in question. His presuppositions allow him no other analysis of his pedagogical failure, and preclude him from experimenting with other methods of presentation.

The extreme form of "postulationism" not only can lead to an educationally disastrous breakdown of communication between specialist and layman; it also has certain intrinsic difficulties. For example, deductions in formal systems abound in combinatorial arguments. If two of the three roots of an equation have been found it is usually taken as a fact (not as a deduction from purely arbitrary postulates), that there is one more. The number of arrangements of three objects is taken as six, without stating that this is merely a valid deduction in a certain postulational system. The group of symmetries of a square is an eight group and is not cyclic. . . . If it is argued that all results of this kind should be labeled "metamathematical" then our thesis can also be restated; it is that a philosophy of mathematics is needed, for undergraduates, in which metamathematics finds a more natural and more significant place.

Let us review, briefly, the reasons for adopting the abstract postulational method. The historical development of this method was influenced by the realization that it is frequently found more efficient to study a system which has been formalized and divorced from meaning. In addition to arguments of efficiency, however, there is a more compelling reason. It is simply the fact that, in a logical system, it is impossible to prove each statement; since such a proof would require the use of still other statements. This argument, we repeat, is based on the decision to define mathematics as a pure logical system. It is our thesis that, from the view of the secondary teacher and the teacher of beginners in college, this decision is neither necessary nor wise. The desirable alternative seems to be to adopt the solution of the scientists: to study mathematics as a system partly logical and partly empirical. This solution is far less drastic than it first appears to be, due to the highly specialized nature of mathematical subject matter. As an illustration of this specialized subject matter and its mathe-

matical analysis let us consider, not "numbers as things in themselves," but number operations, such as addition.

In prehistoric times numbers doubtless originated from tally marks, and the operation of adding 3 and 4 to get 7 was, at first, a symbolic statement of a fairly general scientific fact—a fact, however, not about tally marks, but about a certain physical operation on objects "named" by the tallies. That is to say, while the nature of the tally marks is unimportant, (philosophical researches into the meaning of number are irrelevant here), the operation "+" has a real physical meaning and from this viewpoint $3+4=7$ is empirically verifiable.* The symbolic language need not deceive us; $3+4=7$ is a scientific fact about many objects of experience, and like most scientific generalizations must be applied properly. In an air compressor 3 quarts of air and 4 quarts of air may give 1 quart of air.

Any science proceeds from its basic facts by using induction. Not all possible cases of the laws of physics can be tested, so inductive generalizations are made, and experimentally unprovable assumptions such as the "uniformity of nature" or the "conservation of energy." Mathematics follows the same general procedure. For small "numbers" (*i.e.*, sets of objects), the associative laws, the distributive law, *etc.*, can be proved empirically, but it is more difficult to prove in this way that $10^{1000}+10^{1000}=2\cdot 10^{1000}$. Hence suitable assumptions about the behavior of all "positive integers" are made, the most striking of which is, perhaps, that their generation is a certain type of never-ending process.

In addition to the operations of "addition and multiplication of integers" modern mathematics studies, of course, many other pairing processes; for example, addition modulo 2, in which 1 and 1 are paired with 0; the logical sum of classes; addition of endomorphisms of a ring; *etc.* These and other considerations lead to a definition of mathematics as the science which studies all possible types of pairings, (relations, operations, correspondences). In practice the restriction seems to be added, by tacit consent, that a study is not to be called mathematics unless it is conducted with the aim of analyzing the logical structure and implications of the pairing operations. When, in theoretical mechanics, the pairing of force with mass and acceleration is extensively studied, but with a somewhat different aim, the result is likely to be dubbed "physics" or perhaps "applied mathematics." Furthermore, there seems to be a tendency not to classify a subject as mathematics unless its experimental component (a) is very small or indirect, as in the theory of irrational numbers, and in the traditional concept of geometry, or (b) involves only the most elementary and common experiences of mankind, so that actual experiments can be replaced by "thought experiments," as in the theory of permutations or the theory of probability.

Several of the areas of study just mentioned illustrate the point that the

* In the addition of perpendicular components of vectors, $3+4$ does not equal 7, but 5; but this is merely one of the many additional meanings which has been given to "+" in the course of time, and which we need not consider here.

"elementary" portion, at least, of mathematics is an intimate mixture of nature and man, a blend of mental power and physical reality. Its basic ideas have grown out of experience. Such concepts as number, distance, position, recurrence, continuity, are probably man's earliest, and, in a sense, deepest insights into the orderliness of events; his most basic scientific achievements. Their importance is reflected in the fact that the branches of mathematics which deal with these concepts are growing in number, with lattice theory, the study of metrics, the algebra of classes, the study of continuity, and many others, having been recently added as analyses of the concepts of order, distance, belonging to one or more sets, *etc.* It is significant that in each of these studies experiment alone is inadequate; that the analysis is conducted in terms of (pairing) operations and formally based on postulates. This process is one of the important methods of scientific analysis; it seems to be related to the mental ability to form "models" or "idealizations" from experience. Engineers lean heavily on the idealizations known as "rigid bodies," "particles," *etc.*; and is not the set of symbols, "one, two, three," *etc.*, itself an idealization of a set of similar but distinct objects or events?

To return to our clients. If we adopt the viewpoint toward mathematics which is implicit in the definition above, we need not try to persuade them to reject, in mathematics, all sense evidence such as that used extensively in the primary grades. Instead, we can demonstrate, even to beginners, the necessity for some assumptions about the behavior of large numbers, and thus lay the foundation for critical thinking. We can show them that in some branches of mathematics the "postulational component" is of less importance, (as in the arithmetic of small numbers, or in the theory of finite groups, reckoning with Cayley's Theorem); whereas in other branches it is found fruitful to study the consequences of postulates without regard, at the moment, to the question of whether any entities exist which can be paired in the ways specified by the postulates. We can explain that, by their very nature, the postulates of geometry are not susceptible of a laboratory proof, and that this is a sufficient reason, even if there were no others, for studying the consequences of other sets of geometric assumptions.

Perhaps this "compromise view" of the nature of mathematics will help us to re-establish some needed rapport with our "public," (which is puzzled by the boast that mathematics is now completely divorced from reality), especially if we point out that physics and chemistry, for example, with their theories of "point particles," "frictionless machines," "ideal gases," *etc.*, also have a strong postulational component.

THE ALGEBRA OF SEMI-MAGIC SQUARES

L. M. WEINER, DePaul University

1. Introduction. A semi-magic square is a square array of n^2 numbers for which the sum of the n numbers in any row or column is a constant S. If the sum of each diagonal is also equal to S, the array is referred to as a magic square. The objects referred to as numbers will, for the purposes of this paper, be taken to be the elements of a field F whose characteristic is prime to n. Any such semi-magic square may be considered as an n by n matrix and will hereafter be called an S-matrix of order n according to the following

DEFINITION. *An S-matrix $A=(a_{ij})$ of order n is an n by n matrix for which $\sum_{i=1}^{n} a_{ij}=\sum_{i=1}^{n} a_{ji}=S(A)$ for every j.*

THEOREM 1. *The set of all S-matrices of order n forms a subalgebra R_n of the total matric algebra of degree n which satisfies the following conditions:*

(i) $$S(A+B)=S(A)+S(B),$$

(ii) $$S(kA)=kS(A) \qquad \text{for } k \text{ in } F,$$

(iii) $$S(AB)=S(A)S(B).$$

Proof. Let $A=(a_{ij})$ and $B=(b_{ij})$ be S-matrices of order n, and $C=(c_{ij})=A+B$. Then $c_{ij}=a_{ij}+b_{ij}$, and

$$\sum_{i=1}^{n} c_{ij}=\sum_{i=1}^{n}(a_{ij}+b_{ij})=\sum_{i=1}^{n} a_{ij}+\sum_{i=1}^{n} b_{ij}=S(A)+S(B).$$

The same holds for the rows of C. The matrix kA has elements ka_{ij}, and

$$\sum_{i=1}^{n} ka_{ij}=k\sum_{i=1}^{n} a_{ij}=k\sum_{i=1}^{n} a_{ji}=kS(A)$$

for every j. Next let $D=(d_{ij})=AB$. Then

$$d_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj},$$

and

$$\sum_{i=1}^{n} d_{ij}=\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{kj}=\sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik}b_{kj}=\sum_{k=1}^{n} b_{kj}\sum_{i=1}^{n} a_{ik}=\sum_{k=1}^{n} b_{kj}S(A)=S(B)S(A).$$

A similar computation holds for the rows of D.

Unfortunately, this result cannot be extended to magic squares as is shown by the following counterexample:

$$A = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{bmatrix}, \qquad AB = \begin{bmatrix} 67 & 67 & 91 \\ 67 & 91 & 67 \\ 91 & 67 & 67 \end{bmatrix}.$$

The set of all magic squares of order n, however, does form a linear subspace of the algebra R_n.

The remainder of this paper will be devoted to the determination of the structure of the algebra R_n.

2. The decomposition theorem. Unless otherwise stated, the order of the matrices comprising the algebra R_n will be considered fixed throughout this section, and so the subscript n will be dropped in this and the following section.

We shall first exhibit two ideals of the algebra R. Let M be the set of all S-matrices A for which the entries a_{ij} are all equal to a fixed number a, and let N be the set of all S-matrices A for which $S(A)=0$. It is easy to see that M and N are both ideals of R. If $A=(a_{ij})=(a)$ belongs to M and $B=(b_{ij})$ belongs to R, then the element in the ith row and jth column of $AB=a\sum_{k=1}^{n} b_{kj}=aS(B)$ which indicates that AB belongs to M. Similarly, BA belongs to M. If, on the other hand, A belongs to N and B belongs to R, then $S(AB)=S(BA)=S(A)S(B)=0$.

Next let A be any matrix belonging to R, and let $B=A-C$ where C is the matrix all of whose entries are equal to $S(A)/n$. Then C belongs to M, $S(C)=nS(A)/n=S(A)$, $S(B)=S(A)-S(C)=0$, and B belongs to N. Since $A=B+C$, we see that any element of R is expressible as the sum of an element of M and an element of N; that is, the algebra R is the sum of the algebras M and N. This sum is actually a direct sum; for if A belongs to both M and N, A has a fixed entry a in each place, $S(A)=na=0$, $a=0$, and A is the zero matrix of R. We have proved

THEOREM 2. *The algebra R is the direct sum of the one dimensional ideal M and the ideal N.*

3. The structure of the algebra N. Define A_{ij} to be the matrix which has the number 1 in the first row and first column, the number -1 in the first row and jth column, the number -1 in the ith row and first column, the number 1 in the ith row and jth column, and zeros elsewhere for $i, j=2, 3, \cdots, n$.

THEOREM 3. *Let the S-matrices A_{ij} be as defined above for $n>1$, and $i, j=2, 3, \cdots, n$. Then the algebra N has dimension $(n-1)^2$, and the matrices A_{ij} form a basis for N.*

Proof. Assume first that we have a relationship of the form $\sum_{i,j=2}^{n} a_{ij}A_{ij}=0$. When $2\leqq p, q\leqq n$, $a_{pq}A_{pq}$ is the only component of this sum which contributes anything to the entry in the pth row and qth column which is zero. It follows that $a_{pq}=0$ for $p, q=2, 3, \cdots, n$, and the A_{ij} are linearly independent.

Next let $B=(b_{ij})$ be any S-matrix belonging to N, and consider the S-matrix

$\sum_{i,j=2}^{n} b_{ij}A_{ij}$. This matrix is equal to B since when $2 \leqq p, q \leqq n$, the element in the pth row and qth column is b_{pq}, the element in the first row and qth column is $-\sum_{i=2}^{n} b_{iq} = b_{1q}$, the element in the pth row and first column is $-\sum_{j=2}^{n} b_{pj} = b_{p1}$, and the element in the first row and first column is $\sum_{j=2}^{n}\sum_{i=2}^{n} b_{ij} = -\sum_{j=2}^{n} b_{1j} = b_{11}$. This shows that the S-matrices A_{ij} span N and completes the proof of the theorem.

ON THE EULER CHARACTERISTIC OF A LIE ALGEBRA

S. I. GOLDBERG, Lehigh University

It is well-known that the Euler characteristic $\chi(G)$ of a compact Lie group G is zero. Using H. Hopf's theorem [2] on the cohomology ring of a compact Lie group Chevalley and Eilenberg [1] have shown that if L is a semi-simple Lie algebra over a field of characteristic 0 then $\chi(L)=0$. In this note we generalize this fact to the case of any Lie algebra over an arbitrary field F, and furthermore, we give an elementary proof of this result.*

Let L be a Lie algebra of dimension n defined over F and denote by $C^q(L)$ $(q>0)$ the space of q-linear alternating functions in L with values in F, that is, $C^q(L)$ is the space of those functions f, henceforth called cochains, having the properties:

(a) f is linear in each argument, and

(b) $f(x_{\pi(1)}, \cdots, x_{\pi(q)}) = (\operatorname{sgn}\pi) f(x_1, \cdots, x_q)$

where π is a permutation of the indices $1, \cdots, q$ and $\operatorname{sgn}\pi$ denotes its sign. For $q=0$, $C^q(L)=F$, by definition. Furthermore, $\dim C^q(L) = \binom{n}{q}$, the binomial coefficient. It is this fact which is the crux of the whole proof.

To each element $f \in C^q(L)$ an element $d_q f \in C^{q+1}(L)$ is defined as follows:

$$(1)\quad (d_q f)(x_1, \cdots, x_{q+1}) = \sum_{i<j} (-1)^{i+j+1} f([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{q+1})$$

where $[x_i, x_j]$ denotes the multiplication in the Lie algebra and the 'roof' over an element $x \in L$ implies omission of the argument x. For $q=0$, we set $d_q f = 0$. Using induction, it follows easily that $d_{q+1} d_q f = 0$ for $f \in C^q(L)$ and $q = 0, 1, \cdots, n$.

A cochain f is a 'cocycle' provided $d_q f = 0$. The cocycles of dimension q form

---

* The author wishes to thank the referee for pointing out that the Euler characteristic of 'any' Lie algebra over F vanishes. Originally, the author gave an elementary proof of this fact in the case of a semi-simple Lie algebra over a field of characteristic zero, by making continual use of the full reducibility property of the representation space of a semi-simple Lie algebra.

a (linear) subspace $Z^q(L)$ of $C^q(L)$. For $q=0$, $Z^0=C^0=F$. A cochain $f\in C^q(L)$ is a 'coboundary' if it can be expressed in the form $d_{q-1}g$ for some $g\in C^{q-1}(L)$. The coboundaries form a space $B^q(L)\subset Z^q(L)$ and $B^0(L)=(0)$. Also, $B^q(L)=(0)$ for $q>n$. The quotient space $H^q(L)=Z^q(L)/B^q(L)$ is called the 'qth cohomology group' of L.

Definition. Let $R^q=\dim H^q(L)$, that is R^q is the qth Betti number of L. Then $\chi(L)=\sum_{q=0}^{n}(-1)^qR^q$.

The purpose of this paper is to show that $\chi(L)=0$.

All one needs is the algebraic equivalent of the usual Euler relation for a complex (alternating sum of Betti numbers equals alternating sum of numbers of cells of various dimensions), namely: If K is a finite chain (or here cochain) complex; *i.e.* a finite sequence of finite vector spaces $C^0, \cdots, C^n$ and homomorphisms d_q: $C^q\to C^{q+1}$ with $d_{q+1}d_q=0$, then $\sum(-1)^q\dim C^q=\sum(-1)^qR^q$.

Now following [3], p. 214, we note that

$$\dim H^q = \dim Z^q - \dim B^q \tag{2}$$

and

$$\dim B^q = \dim C^{q-1} - \dim Z^{q-1}. \tag{3}$$

Putting (2) and (3) together we obtain

$$\dim H^q = \dim Z^q + \dim Z^{q-1} - \dim C^{q-1}. \tag{4}$$

For $q=0$ this reduces to $\dim H^0=\dim Z^0$, for $q=n+1$ to $0=\dim Z^n-\dim C^n$.

Taking the alternating sum, one verifies immediately the above relation. For the Lie algebra L the numbers $\dim C^k(L)$ are given by $\binom{n}{k}$, as mentioned above. It follows that

$$\sum(-1)^qR^q = \sum(-1)^q\binom{n}{q} = 0,$$

which we wished to prove.

We note that the explicit form of the coboundary operator (the fact that it involves the Lie-bracket) is completely immaterial.

References

1. C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, Trans. Amer. Math. Soc., vol. 63, 1948, pp. 85–124.

2. H. Hopf, Über die Topologie der Gruppen Mannigfaltigkeiten und ihre Verallgemeinerungen, Ann. of Math., vol. 42, 1941, pp. 22–52.

3. P. Alexandroff and H. Hopf, *Topologie*, Berlin, J. Springer, 1935.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

A GENERALIZATION OF THE EULER-FERMAT THEOREM

D. G. DUNCAN, San Jose State College

In this note the well known Euler-Fermat theorem ($a^{\phi(m)} \equiv 1 \pmod{m}$) for all a such that $(a, m) = 1$) is sharpened by showing that the congruence often holds for a modulus $k > m$. The condition $(a, m) = 1$ is replaced by an analogous condition $(a, t) = 1$, where $t \leqq k$.

THEOREM: *Let n be any integer and let $p_1, \cdots, p_r$ be the set of all primes such that $(p_i - 1) \mid n$ for all i, and let $\alpha_1, \cdots, \alpha_r$ be the maximum powers of these primes such that $\phi(p_i^{\alpha_i}) \mid n$; then $a^n \equiv 1 \pmod{p_1^{\alpha_1} \cdots p_r^{\alpha_r}}$, for all a such that $(a, p_1 \cdots p_r) = 1$.*

Proof. We have $a^{\phi(p_i^{\alpha_i})} \equiv 1 \pmod{p_i^{\alpha_i}}$ for $(a, p_i) = 1$. Now since $\phi(p_i^{\alpha_i}) \mid n$, it follows that

$$a^n \equiv 1 \pmod{p_i^{\alpha_i}} \quad \text{for} \quad (a, p_i) = 1.$$

However, since $(p_i, p_j) = 1$ for $i \neq j$, it follows at once from the elementary theory of congruences that

$$a^n \equiv 1 \pmod{p_1^{\alpha_1} \cdots p_r^{\alpha_r}} \quad \text{for} \quad (a, p_1 \cdots p_r) = 1.$$

The case of most interest is that in which there exists an m such that $\phi(m) = n$. For example, the congruence $a^4 \equiv 1 \pmod{12}$ for $(a, 12) = 1$ (Euler's theorem) gives rise to the stronger relation

$$a^4 \equiv 1 \pmod{2^3 \cdot 3 \cdot 5 = 120} \qquad \text{for } (a, 30) = 1.$$

Similarly the congruence

$$a^{60} \equiv 1 \pmod{61} \quad \text{for} \quad (a, 61) = 1$$

gives rise to

$$a^{60} \equiv 1 \pmod{k = 61 \cdot 2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 = 3{,}407{,}203{,}800}$$

$$\text{for } \left(a, \frac{k}{60}\right) = 1.$$

A SPECIAL DETERMINANT

L. CARLITZ, Duke University

Put

$$\Delta_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ 1 & 2^4 & 3^4 & \cdots & n^4 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2^{2n-2} & 3^{2n-2} & \cdots & n^{2n-2} \end{vmatrix}. \tag{1}$$

Shapiro [1] has proved that if $2n-1=p$, where p is a prime, then Δ_n is not divisible by p. This result can be obtained very rapidly as follows. Subtract the $(n-1)$-th column of Δ_n from the n-th column. Since $(n-1)^{2s} \equiv n^{2s} \pmod{p}$ it follows that

$$\Delta_n \equiv (-1)^{n-1} D_n \pmod{p},$$

where

$$D_n = \left| r^{2s} \right|, \qquad (r, s = 1, \cdots, n-1).$$

Since

$$D_n = ((n-1)!)^2 \prod_{1 \le r < s \le n-1} (r^2 - s^2) \not\equiv 0 \pmod{p},$$

the assertion follows at once.

We may observe that when $2n+1=p$, then again Δ_n is not divisible by the prime p. To see this we add the first, second, $\cdots$, $(n-1)$-th column of Δ_n to the n-th column. Since

$$\sum_{r=1}^{p} r = \tfrac{1}{2} n(n+1) \not\equiv 0 \pmod{p},$$

while

$$\sum_{r=1}^{n} r^{2s} \equiv \frac{1}{2} \sum_{r=1}^{p-1} r^{2s} \equiv 0 \pmod{p} \qquad (1 \le s \le n-1),$$

it follows as in the previous case that $\Delta_n \not\equiv 0 \pmod{p}$.

We now show that*

$$\Delta_n = \frac{1!3!5! \cdots (2n-1)!}{(2n-1) \cdot (n-1)!} \tag{2}$$

for arbitrary $n \ge 1$. Indeed expanding Δ_n with respect to the elements of the first row we get

---

* This result was obtained independently but slightly later by Dr. Morris Newman of the National Bureau of Standards.

$$\Delta_n = \sum_{r=1}^{n} (-1)^{r-1} r D_r, \tag{3}$$

where

$$D_r = |i^{2j}| \qquad (i = 1, \cdots, n;\ i \neq r;\ j = 1, \cdots, n-1).$$

Then by a familiar formula

$$\begin{aligned} D_r &= \frac{(n!)^2}{r^2} \prod_{1<i<j<n,\, i\neq r,\, j\neq r} (j^2 - i^2) \\ &= \frac{(n!)^2}{r^2} \cdot \frac{V_n}{(r-1)!(n-r)!} \cdot \frac{1}{(r+1)\cdots(2r-1)(2r+1)\cdots(r+n)} \\ &= \frac{2(n!)^2 V_n}{(n-r)!(n+r)!} \end{aligned} \tag{4}$$

where

$$V_n = \prod_{1\leq i<j\leq n} (j^2 - i^2) = \frac{1!3!5!\cdots(2n-1)!}{n!}. \tag{5}$$

Substituting from (4) in (3) we get

$$\begin{aligned} \Delta_n &= 2V_n \frac{(n!)^2}{(2n)!} \sum_{r=1}^{n} (-1)^{r-1} r \binom{2n}{n-r} \\ &= 2V_n \frac{(n!)^2}{(2n)!} \sum_{r=0}^{n-1} (-1)^{n-r-1} (n-r) \binom{2n}{r} \\ &= 2V_n \frac{(n!)^2}{(2n)!} \left\{ n\binom{2n-1}{n-1} - 2n\binom{2n-2}{n-2} \right\} \\ &= \frac{nV_n}{2n-1}. \end{aligned}$$

Using (5) it is clear that we get (2).

If p is a prime such that $n < p \leqq 2n+1$, it follows at once from (2) that the highest power of p dividing Δ_n is

$$\begin{cases} n - \frac{1}{2}(p-1) & (p < 2n-1), \\ 0 & (p = 2n-1,\ 2n+1). \end{cases} \tag{6}$$

The latter half of (6) evidently includes the two special cases mentioned above.

Reference

1. V. L. Shapiro, A theorem on a special class of near-Vandermonde determinants, This Monthly, vol. 60, 1953, pp. 697–699.

A DIOPHANTINE MATRIX EQUATION

JOHN JONES, JR., Mississippi Southern College

M. H. Ingraham and H. C. Trimble [1] considered the matrix equation $XA=BX+C$. W. V. Parker [2] considered the matrix equation $AX=XB$. W. E. Roth [3] found a solution to the matrix equation $AX^2+BX+XC=D$, and [4] also found a necessary and sufficient condition for a solution to matrix equations of the types $AX-YB=C$, and $AX-XB=C$. The principal purpose of this paper is to prove the following theorem.

THEOREM. *A necessary and sufficient condition that the Diophantine matrix equation* $A(\lambda)X(\lambda)-Y(\lambda)B(\lambda)=C(\lambda)$ *have a solution,* $X(\lambda)$, $Y(\lambda)$, *of n by n matrices with elements which are analytic functions of a single complex variable* λ *in* $\mathfrak{R}$, *where the elements of the n by n matrices* $A(\lambda)$, $B(\lambda)$, *and* $C(\lambda)$ *are also analytic functions of a single complex variable* λ *in* $\mathfrak{R}$, *is that the matrices*

$$\text{(1)} \qquad \begin{pmatrix} A(\lambda) & C(\lambda) \\ 0 & B(\lambda) \end{pmatrix}, \quad \begin{pmatrix} A(\lambda) & 0 \\ 0 & B(\lambda) \end{pmatrix}$$

be equivalent.

If a solution of the above matrix equation $A(\lambda)X(\lambda)-Y(\lambda)B(\lambda)=C(\lambda)$ exists, we have

$$\begin{pmatrix} I & Y(\lambda) \\ 0 & I \end{pmatrix}\cdot\begin{pmatrix} A(\lambda) & C(\lambda) \\ 0 & B(\lambda) \end{pmatrix}\cdot\begin{pmatrix} I & -X(\lambda) \\ 0 & I \end{pmatrix}=\begin{pmatrix} A(\lambda) & C(\lambda)-A(\lambda)X(\lambda)+Y(\lambda)B(\lambda) \\ 0 & B(\lambda) \end{pmatrix}$$
$$=\begin{pmatrix} A(\lambda) & 0 \\ 0 & B(\lambda) \end{pmatrix}$$

where I is the identity matrix; therefore the matrices (1) are equivalent.

Let the matrices (1) be equivalent; we are to show that the matrix equation of the theorem has a solution. J. H. M. Wedderburn [5] has shown that if $A(\lambda)$ is a matrix with elements which are analytic functions of a single complex variable in a region $\mathfrak{R}$, there exist two non-singular matrices $P(\lambda)$ and $Q(\lambda)$ which have elements which are analytic in $\mathfrak{R}$ and are such that

$$P(\lambda)\cdot A(\lambda)\cdot Q(\lambda) = E_1(\lambda) \dotplus E_2(\lambda) \dotplus \cdots \dotplus E_r(\lambda) \dotplus 0 \dotplus \cdots \dotplus 0,$$

where $E_i(\lambda)\,|\,E_{i+1}(\lambda)$, for $(i=1, 2, \cdots, r)$, *i.e.*, $E_i(\lambda)$ are functions which are analytic in $\mathfrak{R}$ and such that $E_s(\lambda)$ is a factor of $E_t(\lambda)$ when $s<t$, $(s, t=1, 2, \cdots, r)$. Choose non-singular n by n matrices $P(\lambda)$, $Q(\lambda)$, $R(\lambda)$ and $S(\lambda)$ with elements which are analytic functions of a single complex variable in $\mathfrak{R}$ such that

$$\text{(1')} \qquad \begin{aligned} P(\lambda)\cdot A(\lambda)\cdot Q(\lambda) &= A'(\lambda) = a_1(\lambda) \dotplus a_2(\lambda) \dotplus \cdots \dotplus a_\alpha(\lambda) \dotplus 0 \dotplus \cdots \dotplus 0, \\ R(\lambda)\cdot B(\lambda)\cdot S(\lambda) &= B'(\lambda) = b_1(\lambda) \dotplus b_2(\lambda) \dotplus \cdots \dotplus b_\beta(\lambda) \dotplus 0 \dotplus \cdots \dotplus 0, \end{aligned}$$

where $a_i(\lambda)$ and $b_j(\lambda)$ are the invariant factors of $A(\lambda)$ and $B(\lambda)$, respectively,

$(i=1, 2, 3, \cdots, \alpha)$, $(j=1, 2, 3, \cdots, \beta)$. Since we have assumed the matrices (1) to be equivalent, we have

$$M(\lambda) = \begin{pmatrix} P(\lambda) & 0 \\ 0 & R(\lambda) \end{pmatrix}\begin{pmatrix} A(\lambda) & 0 \\ 0 & B(\lambda) \end{pmatrix}\begin{pmatrix} Q(\lambda) & 0 \\ 0 & S(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} A'(\lambda) & 0 \\ 0 & B'(\lambda) \end{pmatrix} = \begin{pmatrix} P(\lambda)A(\lambda)Q(\lambda) & 0 \\ 0 & R(\lambda)B(\lambda)S(\lambda) \end{pmatrix},$$

$$N(\lambda) = \begin{pmatrix} P(\lambda) & 0 \\ 0 & R(\lambda) \end{pmatrix}\begin{pmatrix} A(\lambda) & C(\lambda) \\ 0 & B(\lambda) \end{pmatrix}\begin{pmatrix} Q(\lambda) & 0 \\ 0 & S(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} A'(\lambda) & C'(\lambda) \\ 0 & B'(\lambda) \end{pmatrix} = \begin{pmatrix} P(\lambda)A(\lambda)Q(\lambda) & P(\lambda)C(\lambda)S(\lambda) \\ 0 & R(\lambda)B(\lambda)S(\lambda) \end{pmatrix},$$

where $P(\lambda)C(\lambda)S(\lambda)=(c_{ij})$, $(i, j=1, 2, \cdots n)$. Thus the matrices $M(\lambda)$ and $N(\lambda)$ are equivalent. Associated with the matrices of (1) is the matric equation, $A(\lambda)X(\lambda)-Y(\lambda)B(\lambda)=C(\lambda)$. Now, if we show that there exists a solution to a matric equation of the type

$$A'(\lambda)U(\lambda) - V(\lambda)B'(\lambda) = C'(\lambda), \tag{2}$$

associated with the pair of matrices $M(\lambda)$ and $N(\lambda)$ equivalent to (1), then we can obtain a solution to the original equation associated with the matrices of (1), where the elements of $U(\lambda)$ and $V(\lambda)$ are analytic functions of a single complex variable in $\mathfrak{R}$. Consider (2) element-wise,

$$a_i(\lambda)u_{ij}(\lambda) - v_{ij}(\lambda)b_j(\lambda) = c_{ij}(\lambda), \tag{3}$$

where $(i, j=1, 2, 3, \cdots, n)$, and $a_i(\lambda)\in A'(\lambda)$, $u_{ij}(\lambda)\in U(\lambda)$, $b_{ij}(\lambda)\in B'(\lambda)$, $v_{ij}(\lambda)\in V(\lambda)$, and $c_{ij}(\lambda)\in C'(\lambda)$. Consider elements of $C'(\lambda)$ in four cases: case (i) $c_{ij}(\lambda)$ where $(1\leqq i\leqq\alpha)$, $(1\leqq j\leqq\beta)$; cases (ii) and (iii) $c_{ij}(\lambda)$ where $(1\leqq i\leqq\alpha)$, $(\beta<j\leqq n)$ and $(\alpha<i\leqq n)$, and $(1\leqq j\leqq\beta)$; and case (iv) $c_{ij}(\lambda)$ where $(\alpha<i\leqq n)$ and $(\beta<j\leqq n)$.

In case (i), equation (3) has a solution. For any $c_{ij}(\lambda)$, the pair $a_i(\lambda)$, $b_j(\lambda)$ either have a common factor not a non-zero constant or they are relatively prime to each other. If they are relatively prime to each other, then there exist analytic functions in $\mathfrak{R}$, $p_{ij}(\lambda)$ and $q_{ij}(\lambda)$, such that

$$a_i(\lambda)p_{ij}(\lambda) - b_j(\lambda)q_{ij}(\lambda) = 1. \tag{4}$$

By using a lemma due to Wedderburn [5], we may expand $1/a_i(\lambda)b_j(\lambda)$ in terms of its principal parts in a Mittag-Leffler series.

$$1/a_i(\lambda)b_j(\lambda) = F_{ij}(\lambda) + G_{ij}(\lambda) + \phi_{ij}(\lambda), \tag{5}$$

where $F_{ij}(\lambda)$ and $G_{ij}(\lambda)$ are the parts of this series arising from the zeros of $a_i(\lambda)$ and $b_j(\lambda)$ respectively, which lie in $\mathfrak{R}$, and $\phi_{ij}(\lambda)$ is a function which is analytic in $\mathfrak{R}$. Let

$$(6) \qquad p_{ij}(\lambda) = b_j(\lambda)[G_{ij}(\lambda) + \phi_{ij}(\lambda)]; \qquad q_{ij}(\lambda) = -a_i(\lambda)F_{ij}(\lambda).$$

Both $p_{ij}(\lambda)$ and $q_{ij}(\lambda)$ are analytic in $\mathfrak{R}$ and on multiplying both sides of (5) by $a_i(\lambda)b_j(\lambda)$, we have

$$(7) \quad 1 = a_i(\lambda)b_j(\lambda)[G_{ij}(\lambda) + \phi_{ij}(\lambda)] + a_i(\lambda)b_j(\lambda)F_{ij}(\lambda) = a_i(\lambda)p_{ij}(\lambda) - b_j(\lambda)q_{ij}(\lambda).$$

Hence by multiplying (7) by $c_{ij}(\lambda)$ we get

$$(8) \qquad c_{ij}(\lambda) = a_i(\lambda)[p_{ij}(\lambda)c_{ij}(\lambda)] - b_j(\lambda)[q_{ij}(\lambda)c_{ij}(\lambda)],$$

where $(i=1, 2, \cdots, \alpha)$, $(j=1, 2, 3, \cdots, \beta)$, and therefore there exists a solution for any $c_{ij}(\lambda)$ which is analytic in $\mathfrak{R}$. By making use of a technique due to Roth [4] of using determinantal divisors of $N(\lambda)$, it is seen in cases (ii), (iii), and (iv) above that $c_{ij}(\lambda) = z_{ij}(\lambda)[a_i(\lambda), b_j(\lambda)]$, hence equation (3) always has a solution. Thus there exist matrices $U(\lambda)$ and $V(\lambda)$ of order n with elements which are analytic functions of a single complex variable λ in $\mathfrak{R}$ and satisfy equation (2). Upon substituting (1′) in (2) we have,

$$(9) \qquad X(\lambda) = Q(\lambda)U(\lambda)S^{-1}(\lambda), \quad \text{and} \quad Y(\lambda) = P^{-1}(\lambda)V(\lambda)R(\lambda).$$

The theorem is also true when the elements of the matrices $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$ belong to a principal ideal ring $\mathcal{P}$. The necessity is proved as in the preceding theorem. To prove the sufficiency we make use of a theorem due to C. C. MacDuffee [6] which states that every matrix A of rank α with elements in a principal ideal ring $\mathcal{P}$ is equivalent to a diagonal matrix $h_1, h_2, \cdots, h_\alpha, 0, \cdots, 0$ where $h_i \neq 0$ and $h_i \mid h_{i+1}$. The h_i are invariant up to a unit factor in the principal ideal ring $\mathcal{P}$, so that the diagonal form of A is unique if the h_i belong to a set of non-associates in $\mathcal{P}$. Thus there exist unimodular matrices P, Q, R, and S with elements in $\mathcal{P}$ such that

$$PAQ = \text{diagonal matrix } [a_1, a_2, a_3, \cdots a_\alpha, 0, \cdots 0] = A',$$
$$RBS = \text{diagonal matrix } [b_1, b_2, b_3, \cdots, b_\beta, 0, \cdots 0] = B',$$

where A is of rank α and B is of rank β. In a principal ideal ring $\mathcal{P}$ every element neither zero nor a unit can be factored uniquely (except for unit factors) into a product of primes according to MacDuffee [6]. Thus by making use of the same type of argument as that given by W. E. Roth in [4] we obtain a solution (9) to the matrix equation up to a unit factor in $\mathcal{P}$. However, if the elements of A' and B' belong to a set of non-associates, then X and Y will be unique in $\mathcal{P}$ for a given P, Q, R and S which reduce A and B to diagonal form.

References

1. Ingraham, M. H. and Trimble, H. C., On the matrix equation $TA = BT + C$, Amer. Jour. of Math., vol. 63, 1941, pp. 9–28.

2. Parker, W. V., The matrix equation $AX = XB$, Duke Math. Jour., vol. 17, 1950, pp. 43–51.

3. Roth, W. E., The matrix equation $X^2 + AX + XB + C = 0$, Proc. Amer. Math. Soc., vol. 1, 1950, pp. 586–589.

4. ———, The equations $AX-YB=C$, and $AX-XB=C$ in matrices, Proc. Amer. Math. Soc. Vol. 3, 1952, pp. 392–396.

5. Wedderburn, J. H. M., On matrices whose coefficients are functions of a single variable, Trans. Amer. Math. Soc., vol. 11, 1915, pp. 328–332.

6. MacDuffee, C. C., Matrices with elements in a principal ideal ring, Bull. Amer. Math. Soc. Ser. 2, vol. 39, 1933, pp. 564–584.

A PAIR OF MATRICES WITH PROPERTY *P*

HANS SCHNEIDER, Queen's University, Belfast

A set $A_1, \cdots, A_s$ of $n\times n$ matrices with coefficients in an algebraically closed field is said to have property P if there exists an ordering $\alpha_i^{(1)}, \cdots, \alpha_i^{(s)}$, $i=1, \cdots, n$, of the characteristic roots of $A_1, \cdots, A_s$ for which the characteristic roots of any polynomial $p(A_1, \cdots, A_s)$ are $p(\alpha_i^{(1)}, \cdots, \alpha_i^{(s)})$, $i=1, \cdots, n$. In 1936 McCoy [3] proved that the set $A_1, \cdots, A_s$ has property P if and only if every matrix of the form $(A_iA_j-A_jA_i)R$, where R is a polynomial in the A_i, is nilpotent (for an elementary proof see Drazin, Dungey and Gruenberg [2]). More recently two very special cases of this theorem have been proved separately. Thus in 1950 Parker [5] showed that if $AB=B^2=0$, then the characteristic roots of $A+B$ are the same as those of A. (The characteristic roots of B are all zero.) In 1953 Perfect [6], completing a theorem of A. Brauer [1], showed that if $(C-\lambda I)v=0$, and B is a matrix of rank 1 all of whose columns are multiples of the column vector v, then the characteristic roots of $C+B$ are obtained from those of C by replacing one λ by $\lambda+\text{trace } B$. (One characteristic root of B is trace B, the rest are zero, and $AB=0$ if $A=(C-\lambda I)$.) We shall give a very simple proof of a special case of McCoy's theorem which includes the two results quoted above.

In order to make our theorem applicable to $n\times n$ matrices with coefficients in a field which is not necessarily algebraically closed we shall state the result in terms of the characteristic polynomial $|xI-A|$ of a square matrix A.

LEMMA. *Let $A_1, \cdots, A_s$ be a set of $n\times n$ matrices with coefficients in k. If $(\sum_{i=1}^{j} A_i)A_{j+1}=0$, for $j=1, \cdots, s-1$, then the characteristic polynomial of $\sum_{j=1}^{s} A_j$ is $(\prod_{j=1}^{s} |xI-A_j|)/x^{(s-1)n}$.*

Proof: Since $A_1A_2=0$,

$$\begin{aligned} |xI-A_1|\,|xI-A_2| &= |x^2-x(A_1+A_2)+A_1A_2| \\ &= |x(xI-(A_1+A_2))| = x^n|xI-(A_1+A_2)|. \end{aligned}$$

The result follows if $s=2$. The lemma is now obtained by induction on s.

THEOREM. *Let A and B be $n\times n$ matrices with coefficients in k. If $AB=0$, then the characteristic polynomial of the polynomial $p(A, B)$ without constant term is*

$$|xI-p(A, 0)|\cdot|xI-p(0, B)|/x^n.$$

Proof. Since $AB=0$, we have $p(A, B)=A_1+A_2+A_3$, where $A_1=p(A, 0)$,

$A_3=p(0, B)$ and A_2 is of form $Bq(A, B)A$. Clearly $A_2^2=0$, and so the characteristic polynomial of A_2 is x^n. As $A_1A_2=(A_1+A_2)A_3=0$, it follows by the lemma that

$$\begin{aligned}|xI-p(A, B)| &= |xI-A_1|\,|xI-A_2|\,|xI-A_3|/x^{2n}\\ &= |xI-p(A, 0)|\,|xI-p(0, B)|/x^n.\end{aligned}$$

COROLLARY. *If k is algebraically closed, then the pair of matrices A, B has property P.*

Proof. If α_i, β_i, $i=1, \cdots, n$, are the characteristic roots of A and B respectively then $x^n|xI-(A+B)| = |xI-A|\,|xI-B| = \prod_{i=1}^n (x-\alpha_i)(x-\beta_i)$. We may easily show from this that there is an ordering α_i, β_i, $i=1, \cdots, n$, for which $\alpha_i\beta_i=0$, $i=1, \cdots, n$. It is enough to prove that with this ordering the characteristic roots of $p(A, B)$ are $p(\alpha_i, \beta_i)$, $i=1, \cdots, n$, for any polynomial $p(A, B)$ without constant term.

For such a polynomial, $p(\alpha_i, \beta_i)=p(\alpha_i, 0)+p(0, \beta_i)$ and $p(\alpha_i, 0)p(0, \beta_i)=0$, $i=1, \cdots, n$.

By the theorem,

$$\begin{aligned}x^n|xI-p(A, B)| &= |xI-p(A, 0)|\,|xI-p(0, B)|\\ &= \prod_{i=1}^n (x-p(\alpha_i, 0))(x-p(0, \beta_i))\\ &= x^n\prod_{i=1}^n (x-p(\alpha_i, \beta_i)).\end{aligned}$$

The corollary is proved.

We remark that if $AB=0$ then $((AB-BA)R)^2=0$, where R is any polynomial in A and B. Thus we have indeed proved a special case of McCoy's theorem. A slight extension of the arguments used in the theorem and corollary leads to the following results: Let $A_1, \cdots, A_s$ be a set of $n\times n$ matrices with coefficients in k such that $A_iA_j=0$ if $i<j$. If $p(A_1, \cdots, A_s)$ is a polynomial without constant term, then the characteristic polynomial of $p(A_1, \cdots, A_s)$ is

$$\left(\prod_{j=1}^s |xI-p_j(A_j)|\right)/x^{(s-1)n},$$

where $p_1(A_1)=p(A_1, 0, \cdots, 0)$, *etc.* If in addition, k is algebraically closed, then the set $A_1, \cdots, A_s$ has property P. It may also be shown that if $A_1, \cdots, A_s$ satisfy the conditions of the lemma and k is algebraically closed, then the characteristic roots of $\sum_{j=1}^s A_j$ are $\sum_{j=1}^s \alpha_i^{(j)}$, $i=1, \cdots, n$, for an ordering $\alpha_i^{(1)}, \cdots, \alpha_i^{(s)}$, $i=1, \cdots, n$, of the characteristic roots of $A_1, \cdots, A_s$. This is a property weaker than property L (*cf.* Motzkin and Taussky [4]).

I am indebted to a referee for materially simplifying the proof of the lemma.

References

1. A. Brauer, Limits for the characteristic roots of a matrix, IV. Duke Math. J., vol. 19, 1952, pp. 75–91.
2. M. P. Drazin, J. W. Dungey and K. W. Gruenberg, Some theorems on commutative matrices, J. London Math. Soc., vol. 26, 1951, pp. 221–228.
3. N. H. McCoy, On the characteristic roots of matrix polynomials, Bull. American Math. Soc., vol. 42, 1936, pp. 592–600.
4. T. S. Motzkin and O. Taussky, Pairs of matrices with property L. Trans. American Math Soc., vol. 73, 1952, pp. 108–114.
5. W. V. Parker, On matrices whose characteristic equations are identical. Proc. American Math. Soc., vol. 1, 1950, pp. 464–466.
6. H. Perfect, Method of constructing certain stochastic matrices. Duke Math. J., vol. 20, 1953, pp. 395–404.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

AREA OF A TRIANGLE AND THE DETERMINANT

G. M. PETERSEN, University of Oklahoma

It is easy to demonstrate to a class that the equation of a straight line through (x_2, y_2) and (x_3, y_3) is given by

$$\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

The altitude through (x_1, y_1) of a triangle with vertices (x_1, y_1) (x_2, y_2) and (x_3, y_3) is found in the usual way from the normal form of the above:

$$\frac{1}{\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Another step shows the area of the triangle to be

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

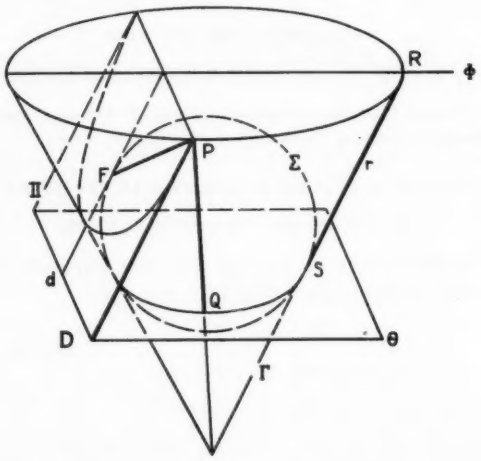
This exposition reveals the *modus operandi* more clearly than the usual proof of showing a collection of terms to be identical with a determinant.

ON DEFINING CONIC SECTIONS

G. B. HUFF, The University of Georgia

The conventional American text in analytic geometry begins the study of conic sections with a paragraph describing these curves as sections of a right circular cone by suitably chosen planes. It is customary to remark that this definition of a conic section is not adapted to the methods and objectives of the course, and each one of the parabola, the ellipse, and the hyperbola is then given its own definition as a locus in a later paragraph. Ordinarily there is no attempt to give even an intuitive proof of the equivalence of the two definitions.

Apparently it is not well known that this may be done quickly and simply in the case of the ellipse and the hyperbola. In *Anschauliche Geometrie* (Hilbert, Cohn-Vossen; Dover; 1944; pp. 7, 8), a figure is given from which a simple



intuitive argument may be made that the two definitions for the ellipse (or the hyperbola) are equivalent. This note is written to call these figures to the attention of analytic geometry teachers and to give the construction of a third figure which leads to a demonstration of the equivalence of the two definitions of the parabola.

Let the parabola be defined as the intersection of a right circular cone Γ and a plane Π parallel to a ruling r of Γ and perpendicular to the plane on r and the axis of Γ. Construct a sphere Σ internally tangent to Γ and tangent to the plane Π. (The existence and unicity of Σ is made clear by thinking of fitting a small balloon in Γ and inflating it until it meets Π.) Let Θ be the plane determined by the intersection of Σ and Γ. Let F be the common point of Π and Σ,

and let d be the common line of Π and Θ. I claim that F is the focus and d is the directrix of the parabola defined by Γ and Π.

Let P be any point of the parabola and let Φ be the plane through P and perpendicular to the axis of the cone Γ. Let D be the foot of the perpendicular from P to d; let Q be the intersection of the ruling on P with the plane Θ, and let R, S be the points on r cut out by Φ and Θ. A simple argument shows that RS is parallel to PD.

Since PF and PQ are tangents to the sphere, then $PF=PQ$.

Since PQ and RS are segments of rulings of Γ cut out by the parallel planes Θ and Φ, then $PQ=RS$.

Since PD is parallel to RS, these segments are cut out on parallel lines by parallel planes and $RS=PD$. It follows that $PF=PD$, as claimed.

It is easy to generalize the figure and the argument to definition of a conic of eccentricity e in terms of its directrix and focus. (Zwikker, *Advanced Plane Geometry*, Amsterdam, 1950, p. 108.)

A DIOPHANTINE EQUATION CHARACTERIZING THE LAW OF COSINES

I. A. Barnett, University of Cincinnati

The purpose of this remark is to show that the most general rational solution of the cubic equation

$$x^2 + y^2 + z^2 + 2xyz = 1 \tag{1}$$

is given by

$$x = \frac{b^2 + c^2 - a^2}{2bc}, \quad y = \frac{a^2 + c^2 - b^2}{2ac}, \quad z = \frac{a^2 + b^2 - c^2}{2ab}. \tag{2}$$

In the first place it is seen at once that equation (1) may be written in the form

$$\begin{vmatrix} y & x & -1 \\ -1 & z & y \\ z & -1 & x \end{vmatrix} = 0.$$

Hence there exist rational numbers a, b, c such that

$$bx + ay = c$$
$$cy + bz = a$$
$$cx + az = b.$$

From these equations we obtain the solution (2). Conversely, the expressions in (2) satisfy (1).

Equation (1) has an immediate geometrical interpretation. For, if a semicircle of radius 1 is divided into three parts subtending angles of magnitude 2α, 2β, 2γ at the center, with the corresponding opposite chords of lengths

$2x, 2y, 2z$ respectively, we have $\alpha+\beta=90°-\gamma$ so that $\cos(\alpha+\beta)=\sin\gamma$ and hence, $\sqrt{(1-x^2)(1-y^2)}=xy+z$. When this is freed of radicals we obtain (1).

This interpretation suggests a generalization of the law of cosines. Dividing the semicircle of radius 1 into n angles $2\alpha_1, 2\alpha_2, \cdots, 2\alpha_n$, and designating the opposite chords by $2x_1, 2x_2, \cdots, 2x_n$, we have immediately $\cos(\alpha_1+\alpha_2\cdots+\alpha_n)=0$. The resulting equation when freed of radicals will be symmetric in $x_1, x_2, \cdots, x_n$.

However, even for $n=4$ the equation will be of the sixteenth degree in x_1, x_2, x_3 and x_4. In fact if $\alpha+\beta+\gamma+\delta=90°$ we see that $\cos(\alpha+\beta)=\sin(\gamma+\delta)$.

Hence

$$\sqrt{(1-x^2)(1-y^2)}-u\sqrt{1-z^2}=xy+z\sqrt{1-u^2},$$

where we have called the lengths of the opposite chords $2x$, $2y$, $2z$, $2u$. When this equation is freed of radicals we obtain

$$[(1-x^2-y^2-z^2-u^2)^2-4(x^2y^2z^2+x^2y^2u^2+x^2z^2u^2+y^2z^2u^2-2x^2y^2z^2u^2)]^2$$
$$=64x^2y^2z^2u^2(1-x^2)(1-y^2)(1-z^2)(1-u^2).$$

In terms of the elementary symmetric functions of x^2, y^2, z^2, u^2,

$$r=\sum x^2, \qquad s=\sum x^2y^2, \qquad t=\sum x^2y^2z^2, \qquad w=x^2y^2z^2u^2$$

the preceding equation may be written

$$[(1-r)^2-4(t-2w)]^2=64w(1-r+s-t+w).$$

It is not clear how one would go about solving such a diophantine equation.

ON THE $\lim_{\theta\to 0}\cos\theta$

M. J. PASCUAL, Siena College

In most texts on calculus, the proof of the fundamental limit equation $\lim_{\theta\to 0}\sin\theta/\theta=1$ involves the assumption that the limit equation $\lim_{\theta\to 0}\cos\theta=1$ is evident. The apparent reason for this supposedly obvious conclusion must have its basis in the fact that $\cos\theta$ is continuous at $\theta=0$. But the notion of limit is a more primitive one than that of continuity. Hence to avoid the possible confusion which may easily arise in the mind of a beginner in calculus, the teacher should at least assert that the $\lim_{\theta\to 0}\cos\theta$ could be proved to be 1 by direct appeal to the definition of limit. Indeed, we could give the following proof, which would simultaneously introduce the student to the analytical method of proof and also prepare an example for the lesson on continuity, in which the instructor shows that $\cos\theta$ is actually continuous at $\theta=0$. Perhaps it could be left to the students to show it is continuous at any value of θ.

Since this proof would be given at an early stage of a calculus course, prior to which inverse trigonometric functions may not have been covered, it will be desirous to avoid such functions, so we shall define $\cos\theta$ by the conventional

method employing rectangular coordinates. This will also prove helpful in establishing the continuity at $\theta=0$, as the old triangle definition will not suffice, there being no triangle.

Placing the vertex of θ at the origin of the rectangular system, with its initial side along the positive x-axis, and measuring θ as usual counterclockwise, we define $\cos\theta$ by choosing an arbitrary point other than (0, 0) on the terminal side of θ having coordinates (x, y) and let $\cos\theta=x/\sqrt{x^2+y^2}$. Since $y\to 0$ as $\theta\to 0$, we wish to establish the equation $\lim_{y\to 0} x/\sqrt{x^2+y^2}=1$. We shall restrict θ to the first or fourth quadrants and for convenience we may choose $x=1$.

To fulfill the definition of limit, for any $\epsilon>0$, we must produce a δ_ϵ such that

$$\left|\frac{1}{\sqrt{1+y^2}}-1\right|<\epsilon \quad \text{for} \quad |y|<\delta_\epsilon.$$

We analyze as follows: for any $y\neq 0$

$$0<\frac{1}{\sqrt{1+y^2}}<1 \quad \text{so that} \quad \left|\frac{1}{\sqrt{1+y^2}}-1\right|=1-\frac{1}{\sqrt{1+y^2}}.$$

Hence we wish to have

$$1-\frac{1}{\sqrt{1+y^2}}<\epsilon$$

and for $0<\epsilon<1$ (for $\epsilon\geqq 1$ this inequality obviously holds)

$$\begin{aligned} 1-\epsilon &< \frac{1}{\sqrt{1+y^2}} \\ 1+y^2 &< (1-\epsilon)^{-2} \\ |y| &< \sqrt{(1-\epsilon)^{-2}-1} \end{aligned}$$

giving the desired δ_ϵ as

$$\sqrt{(1-\epsilon)^{-2}-1}.$$

A DE-CENTRALIZED CENTROID

J. P. Ballantine, University of Washington

My students are always interested to hear about a region in Quadrant I whose centroid is on the x-axis. This is, of course, slightly impossible. Moreover, this particular region is wide at one end, and tapers out to a sharp point, with the centroid at the end of the point. This is getting more impossible. Finally, the height of impossibility is reached with the statement that the sharp point reaches out infinitely far.

The region is that included between $y=1/x$ and $y=0$, from $x=1$ to $x=\infty$.

THE PERIOD OF POLAR CURVES

R. LARIVIERE, University of Illinois, Chicago

Many calculus teachers find students in their classes who are not aware that the period of the polar curves of many trigonometric functions is not the same as that of the corresponding curve in rectangular coordinates.

An interesting way of enlightening these students is to teach them generalized polar coordinates (if they have not already learned them) and to point out that the curve cannot become periodic until the points indicated by increasing values of θ have the same coordinates as those of points already on the curve.

Thus, the generalized coordinates of a point (ρ, θ) on the curve $\rho = f(\theta)$ are $[(-1)^r\rho, (\theta + r\pi)]$, where r is any integer. Since we are concerned with increasing values of θ only we limit our consideration to $r > 0$. When $f(\theta + r\pi) = (-1)^r f(\theta)$ for all values of θ the curve repeats, and the minimum $r\pi$ at which this occurs is the polar period R.

Examples.

Curve	$f(\theta + r\pi)$	$(-1)^r f(\theta)$	R
$\rho = \sin 5\theta$	$\sin(5\theta + 5r\pi)$	$(-1)^r \sin 5\theta$	1π
$\rho = \cos\frac{3\theta}{4}$	$\cos\left(\frac{3\theta}{4} + \frac{3r\pi}{4}\right)$	$(-1)^r \cos\frac{3\theta}{4}$	8π
$\rho = \tan\frac{\theta}{3}$	$\tan\left(\frac{\theta}{3} + \frac{r\pi}{3}\right)$	$(-1)^r \tan\frac{\theta}{3}$	6π
$\rho = 1 + \sin\theta$	$1 + \sin(\theta + r\pi) = 1 + \sin\theta\cos r\pi$	$(-1)^r(1 + \sin\theta)$	2π

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1161. *Proposed by C. S. Ogilvy, Hamilton College*

Professor E. P. B. Umbugio has recently been strutting around because he hit upon the solution of the fourth degree equation which results when the radicals are eliminated from the equation

$$x = (x - 1/x)^{1/2} + (1 - 1/x)^{1/2}.$$

Deflate the professor by solving this equation using nothing higher than quadratic equations.

E 1162. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find two non-congruent similar triangles having two sides of one equal to two sides of the other.

E 1163. *Proposed by H. Gupta, Hoshiarpur, India*

Find the number of ways in which m similar objects can be distributed among n persons where there is no restriction as to the number of objects that any of them may receive.

E 1164. *Proposed by M. J. Mansfield, Purdue University*

Every rational number in the closed interval $[0, 1]$ appears at least once in the sequence

$$0, 1, 1/2, 1/3, 2/3, 1/4, 2/4, 3/4, 1/5, 2/5, 3/5, 4/5, \cdots.$$

Find the nth, or general, term of this sequence.

E 1165. *Proposed by Andrew Sobczyk, Los Alamos Scientific Laboratory*

A vertex V of a closed polygon C having an odd (even) number of sides is *regular* in case a triangle formed by extending the sides incident on V and having for base a line segment containing the opposite side (vertex) to V circumscribes C. Show that every convex C has at least one regular vertex. (A convex pentagon may have non-regular vertices.)

SOLUTIONS

The Student's Predicament

E 1131 [1954, 568]. *Proposed by L. A. Ringenberg, Eastern Illinois State College*

A student solves the differential equation $(dy/dx)^2 = x^2$ and reports as follows: "General solution $(2y-2c)^2 = x^4$; p-discriminant $x=0$; c-discriminant $x=0$. Since the p-discriminant locus consists of the envelope, cuspidal, and tac loci while the c-discriminant consists of the envelope, cuspidal, and nodal loci and since the line $x=0$ is not an envelope, it ought to be a cuspidal locus. But the general solution consists of the parabolas $y=x^2/2+c$ and $y=-x^2/2+c$, and parabolas don't have cusps. How can there be a cuspidal locus if there are no cusps?" Resolve the predicament.

Editorial Note. All the contributed solutions accounted for a cuspidal locus by pointing out that the solution of the differential equation is a pencil of composite quartic curves, each factorable into a pair of parabolas tangent to one

another along the line $x=0$, whence $x=0$ is a (double) cuspidal locus. This argument seems hardly satisfactory. The theory that a locus which is part of both the p-discriminant locus and the c-discriminant locus, and whose equation does not satisfy the differential equation, is a cusp locus, assumes that the p-equation is irreducible in the real domain. This is not the case here. The locus $x=0$, which appears as a squared factor in the p-discriminant, is a tac locus; there is no cusp locus as generally understood. See W. W. Johnson, *A Treatise on Ordinary and Partial Differential Equations*, 3rd ed., art. 54, pp. 49–50. The student's predicament arose from employing theory concerning the p and c-discriminants in a case where that theory does not apply. The analytical theory concerning the p-discriminant and c-discriminant loci presents grave difficulties.

Three Tangent Circles and Two Equal Arcs

E 1132 [1954, 568]. *Proposed by Leon Bankoff, Los Angeles, California*

A common external tangent of two circles, tangent externally at C, cuts their smallest circumscribed circle in P and Q. The common internal tangent at C intersects the minor arc PQ in E, and the major arc QP in F. PC extended meets the outer circumference in K. Show that arc QK = arc KF.

Solution by the Proposer. The inversion (E, EC^2) transforms the outer circumference into the line PQ; the line PK becomes the circle through P, C, E. Hence K', the inverse of K, lies on the intersection of the line PQ and the circle PCE. It follows that angles CEK' and CPK' are equal, and arc QK = arc KF.

Also solved by Hüseyin Demir and Josef Langr (synthetically) and R. B. Plymale (analytically).

A Theorem on Polynomials

E 1133 [1954, 568]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute*

Show that a necessary and sufficient condition that the imaginary roots of

$$f(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n = 0, \qquad a_0 \neq 0,$$

occur in conjugate pairs is that the numbers $a_0, a_1, \cdots, a_n$ all lie on a common ray emanating from the origin in the complex plane.

Solution by F. D. Parker, Clarkson College of Technology. The theorem should read, "a common line through the origin" rather than "a common ray emanating from the origin."

If the coefficients lie on a common line, $\theta=\theta_1$, through the origin, then writing the coefficients in polar form produces a factor $e^{i\theta_1}$. When this factor is divided out, the coefficients are real and the imaginary roots are in conjugate pairs.

If the imaginary roots occur in conjugate pairs, then an equation with real coefficients can be formed with those roots. Any other equation with the same roots must differ by only a multiplicative constant. Any such constant will place the new coefficients on a common line through the origin.

Also solved by C. N. Campopiano, D. F. Criley, Hüseyin Demir, E. Frank, A. M. Glicksman, Bernard Greenspan, D. S. Greenstein, Virginia Hanly, A. R. Hyde, M. S. Klamkin, D. C. B. Marsh, C. S. Ogilvy, M. J. Pascual, B. E. Rhoades, L. A. Ringenberg, Azriel Rosenfeld, David Sachs, Paul Schaefer, Berthold Schweizer, R. H. Sprague, J. A. Tierney, and the proposer. Late solution by C. F. Pinzka.

Squares and Perfect Numbers

E 1134 [1954, 568]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that a square integer is not a perfect number.

I. *Solution by C. F. Pinzka, Princeton, N. J.* If $N=\Pi p^{2a}$, the sum of the divisors of N is

$$\Pi \frac{p^{2a+1}-1}{p-1}.$$

Since the latter is always odd, it cannot equal $2N$ as required for a perfect number.

II. *Solution by C. D. Olds, San Jose State College.* Euler proved that an odd perfect number must have the form $r^{4k+1}P^2$, where r is a prime of the form $4n+1$. An even perfect number must be of Euclid's type, that is, of the form $2^{p-1}(2^p-1)$ where 2^p-1 is a prime. Thus a square cannot be perfect.

Also solved by Leon Bankoff, R. L. Beinert, Arne Benson, W. E. Briggs, J. E. Darraugh, A. D. Freedman, Phil Freygood, Bernard Greenspan, D. S. Greenstein, Virginia Hanly, Douglas Holdridge, A. R. Hyde, Bernard Jacobson, M. S. Klamkin, Sidney Kravitz, Octave Levenspiel, D. C. B. Marsh, J. D. Miller, Leo Moser, T. F. Mulcrone, J. B. Muskat, F. D. Parker, L. A. Ringenberg, Azriel Rosenfeld, Georgia Smith, D. R. Sudborough, and the proposer. Late solutions by H. W. Becker and C. W. Trigg.

Maximum Value of a Determinant

E 1135 [1954, 568]. *Proposed by Vern Hoggatt, San Jose State College*

If a third order determinant has elements 1, 2, 3, $\cdots$, 9, what is the maximum value it may have?

Solution by F. W. Saunders, Coker College. Let P be the sum of the three positive terms of the determinant and $-N$ the sum of the three negative terms. The maximum value of P is

$$9\cdot 8\cdot 7 + 6\cdot 5\cdot 4 + 3\cdot 2\cdot 1 = 630$$

The minimum value of N is

$$9\cdot 8\cdot 1 + 7\cdot 5\cdot 2 + 6\cdot 4\cdot 3 = 214.$$

The minimum combination for N consistent with P is

$$9\cdot6\cdot1 + 8\cdot5\cdot2 + 7\cdot4\cdot3 = 218.$$

Any change in P would result in lowering that sum by more than 4. Therefore 412 is the maximum value for the determinant, and one form for the determinant is

$$\begin{vmatrix} 9 & 4 & 2 \\ 3 & 8 & 6 \\ 5 & 1 & 7 \end{vmatrix}.$$

Also solved by Julian Braun, M. S. Klamkin, T. F. Mulcrone, J. B. Muskat, C. F. Pinzka, and T. H. Sumner.

F. Stancliff, in *Scripta Mathematica*, Dec. 1954, p. 278, has obtained, by trial and error, determinants which use each of the nine digits once and having values from 0 to 323 (with the exception of 202) and even values from 324 to 412.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4633. *Proposed by W. E. Weissblum, Republic Aviation Corporation, Farmingdale, N. Y.*

An even number, n, of chairs are arranged in a circle and numbered 1 through n. An equal number of people are numbered 1 through n and line up in an arbitrary manner. Prove that they can be seated in such a way as to preserve their order and such that no person's number is the same as that of his chair. What happens if n is odd?

4634. *Proposed by Chih-yi Wang, University of Minnesota*

Let a real number k, $0<k<1$, be given. Define the elliptic integrals

$$A_s = \int_0^{\pi/2} \sin^{2s} t\sqrt{1 - k^2 \sin^2 t}\, dt, \qquad B_s = \int_0^{\pi/2} \frac{\sin^{2s} t}{\sqrt{1 - k^2 \sin^2 t}}\, dt$$

for $s=0, 1, 2, \cdots$. Prove that

$$\text{(a) } B_r = \frac{B_{r-1} - A_{r-1}}{k^2}, \qquad \text{(b) } A_r = \frac{(2rk^2 - 1)A_{r-1} + (1 - k^2)B_{r-1}}{(2r + 1)k^2},$$

for $r=1, 2, 3, \cdots$.

4635. *Proposed by Albert Babbitt, Rutgers University*

Let G be an additive Abelian group which has no element of order 2 and let f be a one to one mapping of G onto G. Then f is the inversion (*i.e.*, $f(x) = -x$) if and only if the following two conditions are satisfied:

I. The neutral element 0 is the only element of G invariant under f.
II. For every element $a \in G$, there exists an element $b=b(a)$ in G such that, for all $x \in G$,

$$f(x - a) = f^{-1}(x + b) - a.$$

4636. *Proposed by Emma Lehmer, Berkeley, California*

Show that, if n is a prime of the form $8m+5$, then

$$\sum_{r=0}^{n-1} \cos \frac{2\pi r^4}{n} = \sqrt{n}.$$

For what other values of n is this true?

4637. *Proposed by W. E. Briggs, University of Colorado*

Let $f(n)$ be the smallest number such that $u_{f(n)}$ is divisible by n, where $u_1=1$, $u_2=1$ and $u_n=u_{n-1}+u_{n-2}$. Prove (or disprove) that for any prime p greater than 2 and for all n,

$$f(p^n) = p^{n-1}f(p).$$

SOLUTIONS

Inequality

4553 [1953, 554; 1954, 720]. *Proposed by D. J. Newman, Republican Aviation Corporation, Farmingdale, N. Y.*

Consider any sequence $\{a_n\}$ of real numbers. Prove

$$\sum_{n=1}^{\infty} a_n \leqq \sqrt{2} \sum_{n=1}^{\infty} \sqrt{\frac{a_n^2 + a_{n+1}^2 + \cdots}{n}}.$$

Comment by R. P. Boas, Jr., Northwestern University. The more precise inequality

$$\sum_{n=1}^{\infty} a_n \leqq (4/3)^{1/2} \sum_{n=1}^{\infty} (r_n/n)^{1/2}, \qquad r_n = \sum_{k=n}^{\infty} a_k^2,$$

was proved by Stechkin [*Mat. Sbornik N.S.* 29(71), 225–232 (1951)]. His proof is as follows. First we observe that

$$\sum_{n=k}^{\infty} \frac{1}{n^2(n+1)^2} = \sum_{n=k}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)^2 = \sum_{n=k}^{\infty} \left\{\int_n^{n+1} \frac{dx}{x^2}\right\}^2$$
$$\leqq \sum_{n=k}^{\infty} \int_n^{n+1} \frac{dx}{x^4} = \int_k^{\infty} \frac{dx}{x^4} = \frac{1}{3k^3}.$$

Using this, we have by Cauchy's inequality

$$\sum_{n=1}^{\infty} a_n = 2\sum_{n=1}^{\infty} \frac{a_n}{n(n+1)} \sum_{k=1}^{n} k = 2\sum_{k=1}^{\infty} k \sum_{n=k}^{\infty} \frac{a_n}{n(n+1)}$$
$$\leqq 2\sum_{k=1}^{\infty} k \left\{\sum_{n=k}^{\infty} a_n^2\right\}^{1/2} \left\{\sum_{n=k}^{\infty} \frac{1}{n^2(n+1)^2}\right\}^{1/2}$$
$$\leqq \frac{2}{\sqrt{3}} \sum_{k=1}^{\infty} (r_k/k)^{1/2}.$$

The constant $(4/3)^{1/2} = 1.1547$ is not the best possible, and the best constant does not seem to be known. By taking $a_1 = x$, $a_2 = 1$, $a_3 = a_4 = \cdots = 0$, one can show that the best constant cannot be less than 1.12.

A Consequence of Fermat's Equation

4571 [1954, 52]. *Proposed by Robert Kissling, University of California, Berkeley*

Suppose $a^p + b^p = c^p$ in relatively prime integers with p a prime greater than 3. If q is any prime dividing $a^2 + ab + b^2$ but not $a + b - c$, show that $(q-1)/6p$ is an integer.

Solution by C. R. Phelps, Rutgers University. We have $a \not\equiv 0 \pmod{q}$, for otherwise q is a common factor of a and b, from $a^2 + ab + b^2 \equiv 0$. Thus $a^{-1} \pmod{q}$ exists, and the hypotheses give

$$1 + B^p \equiv C^p \text{ and } 1 + B + B^2 \equiv 0 \pmod{q},$$

where $B = ba^{-1}$ and $C = ca^{-1}$. From the latter congruence we get $B^3 \equiv 1$, with $B \not\equiv 1$. (Otherwise $a \equiv b$ and $3a^2 \equiv 0$ with $a \not\equiv 0$, so that $q = 3$; but if $q = 3$ and $a \equiv b \pmod{3}$ then either $a \equiv b \equiv 1$ and $c \equiv 2$ or $a \equiv b \equiv 2$ and $c \equiv 1$, and in either case $a + b - c \equiv 0$ contrary to the hypothesis.) Thus 3 is a divisor of $q-1$, from $B^{q-1} \equiv 1$.

Suppose that $(p, q-1) = 1$. Then there exist x, y such that $px + y(q-1) = 1$ with x odd, and since 3 divides $q-1$, $p \equiv x \pmod{3}$. If $p \equiv 1 \pmod{3}$, then $C^p \equiv 1 + B^p \equiv 1 + B \equiv -B^2 \pmod{q}$, since $B^3 \equiv 1$; also $C \equiv C^{px} \cdot C^{y(q-1)} \equiv C^{px} \equiv -B^{2x} \equiv -B^2 \equiv 1 + B \pmod{q}$, contrary to the hypothesis $1 + B \not\equiv C$. Similarly, if $p \equiv 2 \pmod{3}$, then $C^p \equiv 1 + B^p \equiv 1 + B^2 \equiv -B$ and $C \equiv C^{px} \equiv (-B)^x \equiv -B^2 \equiv 1 + B$,

again contrary to the hypothesis. Thus $(p, q-1)\neq 1$, so that p divides $q-1$.

Evidently $q-1$ is even, and therefore $(q-1)/6p$ is an integer.

Note. The condition $a^p+b^p=c^p$ can be replaced by the more general condition $a^p+b^p\equiv c^p \pmod{q}$, without change in the proof. In this less esoteric form examples can be constructed, e.g.: $a=4$, $b=7$, $c=13$, $p=5$, $q=31$.

A Class of Irrational Numbers

4572 [1954, 263]. *Proposed by Paul Erdös, University of Notre Dame*

Let b_k be a sequence of non-negative integers such that

$$\overline{\lim}\ \frac{1}{n}\sum_{k=1}^{n} b_k < \infty.$$

Let $f(n)\to\infty$ denote the number of b_k's with $1\leqq k\leqq n$ for which $b_k>0$. If $\underline{\lim}\ f(n)/n=0$, show that

$$c=\sum_{k=1}^{\infty}\frac{b_k}{2^k}$$

is irrational.

Solution by G. G. Lorentz, Wayne University. Let

$$c_k = b_k + b_{k+1}/2 + b_{k+2}/2^2 + \cdots.$$

If $(b_1+\cdots+b_k)/k\leqq M$, we have

$$\begin{aligned}\frac{c_1+\cdots+c_k}{k} &\leqq \frac{b_1+\cdots+b_k}{k}+\frac{k+1}{2k}\cdot\frac{b_1+\cdots+b_{k+1}}{k+1}\\ &\quad+\frac{k+2}{2^2k}\cdot\frac{b_1+\cdots+b_{k+2}}{k+2}+\cdots \qquad (1)\\ &\leqq \frac{M}{k}\left\{k+\frac{k+1}{2}+\frac{k+2}{2^2}+\cdots\right\}=2M+\frac{2M}{k}\leqq 4M.\end{aligned}$$

We prove that

$$\underline{\lim}\ c_k = 0. \qquad (2)$$

Assume $c_k\geqq q_0>0$ for all large k. Then, since $2c_{k-1}=2b_{k-1}+c_k$, we have $c_k=2c_{k-1}$ for all k with $b_{k-1}=0$. Let s be a fixed positive integer. We see that $c_k\geqq 2^s q_0$ if b_k is preceded by s vanishing terms b_i. The number of k's between 1 and n which do not have this last property does not exceed $sf(n)$ and is therefore less than $n/2$ for an infinity of values of n. For such n, $c_k\geqq 2^s q_0$ is true for at least $n/2$ values of k, $1\leqq k\leqq n$, so that

$$\frac{c_1+\cdots+c_n}{n}\geqq 2^{s-1}q_0,$$

which contradicts (1) if s is sufficiently large, and proves (2).

Now let $c=p/q$ be rational. Then for some integer a, $c=p/q=a/2^{k-1}+c_k/2^k$, so that

$$p2^{k-1} - qa = \text{integer} = qc_k/2,$$

which is a contradiction.

Also solved by Harry Kesten, P. J. Owens and Howard Robbins.

An Improper Integral

4573 [1954, 126]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Show that, if $p=\log(\sqrt{2}+1)$,

$$\int_0^p \frac{x \sinh x}{\sqrt{1-\sinh^2 x}}\,dx = \frac{\pi}{4}\log 2.$$

I. *Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota.* The substitution $\coth x=y$ yields

$$n_1 = \int_0^p \frac{x \sinh x}{\sqrt{1-\sinh^2 x}}\,dx = \frac{1}{2}\int_{\sqrt{2}}^{\infty} \frac{\log\dfrac{y+1}{y-1}}{(y^2-1)\sqrt{y^2-2}}\,dy.$$

Now consider the integral

$$n(\alpha) = \frac{1}{2}\int_{\sqrt{2}}^{\infty} \frac{\log\dfrac{\alpha y+1}{\alpha y-1}}{(y^2-1)\sqrt{y^2-2}}\,dy.$$

Differentiation with respect to α yields

$$n'(\alpha) = \frac{-1}{\alpha^2}\int_{\sqrt{2}}^{\infty} \frac{y\,dy}{(y^2-1)(y^2-\alpha^{-2})\sqrt{y^2-2}}.$$

This integral is easily evaluated by familiar methods and we obtain

$$n'(\alpha) = \frac{\pi}{2}\left[\frac{\alpha}{(\alpha^2-1)\sqrt{2\alpha^2-1}} - \frac{1}{\alpha^2-1}\right].$$

Hence

$$n(\alpha) = \frac{\pi}{4}\log\frac{(\alpha+1)(\sqrt{2\alpha^2-1}-1)}{(\alpha-1)(\sqrt{2\alpha^2-1}+1)}$$

in which the constant of integration vanishes since $n(\alpha)\to 0$ as $\alpha\to\infty$. On noting that $n(\alpha)\to n_1$ as $\alpha\to 1$, we obtain $n_1=(\pi/4)\log 2$ as required.

II. *Solution by the Proposer.* It is not difficult to show that

$$\log\left(\sqrt{1+\sin\theta}+\sqrt{\sin\theta}\right)=\sin\theta+\frac{1}{2}\frac{\sin 3\theta}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{\sin 5\theta}{5}+\cdots$$

for $0\leqq\theta\leqq\pi$. Integrating from 0 to $\pi/2$ we obtain

$$\int_0^{\pi/2}\log\left(\sqrt{1+\sin\theta}+\sqrt{\sin\theta}\right)d\theta=1+\frac{1}{2}\frac{1}{3^2}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^2}+\cdots=\frac{\pi}{2}\log 2,$$

from the result of problem 4483 [1953, 424].

Letting $\sqrt{1+\sin\theta}+\sqrt{\sin\theta}=e^z$ we have the required result.

Also solved by Leonard Carlitz, P. L. Chessin, and Chih-yi Wang.

Coefficients in the Taylor Expansion for a Rational Function

4574 [1954, 126]. *Proposed by H. S. Shapiro, New York University*

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be the power series of a rational function, and let the a_n be rational integers. If $a_n=0$ for $n=(k!+k)$, $k=1, 2, \cdots$, then f is a polynomial.

Solution by John B. Kelly, Michigan State College. According to a recent theorem of C. Lech (Arkiv für Matematik, vol. 2, 1953, pp. 417–421) the indices n for which $a_n=0$, belong, for n sufficiently large, to a finite set of residue classes modulo some integer, r, and every sufficiently large element n of any of these classes is such that $a_n=0$. But for any modulus r, $k!+k$ will lie in each residue class, modulo r, infinitely often, because when $k\geqq r$, $k!+k\equiv k \pmod r$. Hence, for n sufficiently large $a_n=0$; $f(z)$ is a polynomial. The condition that the coefficients be integers is superfluous.

Also solved by Leonard Carlitz.

A Diophantine System

4575 [1954, 126]. *Proposed by the late Joseph Rosenbaum, Hartford, Connecticut*

Find the complete solution in positive integers of the Diophantine system

$$2uv-xy=16, \qquad xv-uy=12.$$

Solution by C. R. Phelps, Rutgers University. Solving the given equations for x and u, we get

$$x=\frac{24v+16y}{2v^2-y^2} \qquad u=\frac{16v+12y}{2v^2-y^2}.$$

Evidently $2v^2-y^2$ is positive. We find also

$$2(12)^2 - (16)^2 = 32 = (x^2 - 2u^2)(2v^2 - y^2),$$

whence $2v^2-y^2$ is a divisor of 32. Let $2v^2-y^2=2^s$, $0 \leqq s \leqq 5$.

(A) If s is odd, say $2k+1$, from $2v^2-y^2=2^{2k+1}$ by successive applications of divisibility by 2 we get

$$v = 2^k v_0, \qquad y = 2^{k+1} y_0, \qquad v_0^2 - 2y_0^2 = 1, \qquad v_0 \text{ odd.}$$

(B) If s is even, say $2k$, we similarly get

$$v = 2^k v_0, \qquad y = 2^k y_0, \qquad y_0^2 - 2v_0^2 = -1, \qquad y_0 \text{ odd.}$$

In either case all solutions v_0, y_0 are obtained from the solutions of the Pell equation $p^2-2q^2=\pm 1$, which are given by

$$2p_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n, \qquad 2\sqrt{2}q_n = (1+\sqrt{2})^n - (1-\sqrt{2})^n.$$

In case (A), v_0 and y_0 are given by p_n and q_n, respectively, for even values of n; and in case (B), v_0 and y_0 are given by q_n and p_n, respectively, for odd values of n. Since

$$(p_n + q_n\sqrt{2})(1+\sqrt{2})^2 = p_{n+2} + q_{n+2}\sqrt{2},$$

we have the recurrence formulas

$$p_{n+2} = 3p_n + 4q_n, \qquad q_{n+2} = 2p_n + 3q_n.$$

We note also that p_i and q_i are both positive if and only if $i \geqq 0$.

The complete solutions are then put together as follows:

$$(A)\ v = 2^k p_n, \quad y = 2^{k+1} q_n, \quad x = 2^{2-k} p_{n+2}, \quad u = 2^{2-k} q_{n+2}, \qquad n \text{ even}$$

$$(B)\ v = 2^k q_n, \quad y = 2^k p_n, \quad x = 2^{3-k} q_{n+2}, \quad u = 2^{2-k} p_{n+2}, \qquad n \text{ odd;}$$

where in both cases $k=0, 1, 2$.

The solutions in integers all less than 100 are:

n	k	x	u	v	y
1	0	40	28	1	1
	1	20	14	2	2
	2	10	7	4	4
2	0	68	48	3	4
2	1	34	24	6	8
	2	17	12	12	16
3	2	58	41	20	28
4	2	99	70	68	96

Also solved by H. D. Block, Monte Dernham, C. V. Gregg, Louisa S. Grinstein, Stephen Lichtenbaum and Joseph D'Atri, T. M. Little, D. C. B. Marsh, R. S. Underwood, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio

COLLEGE TEXT BOOKS

Editorial Note. It has been felt for some time that as complete as possible a list of text books in the important fields of mathematics, taught in most colleges, should be made available. With this in mind, the MONTHLY has endeavored to collect such information, and list all the recent text books covering the fields of Intermediate Algebra, College Algebra, Trigonometry, Analytic Geometry, Unified Freshman Mathematics, Analytic Geometry and Calculus, Calculus, and Differential Equations, together with the author, publisher and the date published. Also the number of pages and price are included. References following the price are to critical reviews which have appeared in this MONTHLY. In some instances short paragraphs are included which describe unusual contents, features, or methods of presentation. These are not intended as critical reviews and the presence or absence of a paragraph has no relation to the quality of the book.

We have made every effort to prepare a complete list, and hope that this has been accomplished. We fear, however, that there may be omissions or errors, and apologize in advance for these. Moreover, we do not wish to recommend any particular books, but hope that the material will be of sufficient help to warrant the entire listing.

Except in a few cases, the books have been published since 1945. They have been classified by subject first, then by date of publication, and within any given year, alphabetically by author.

INTERMEDIATE ALGEBRA

Except where noted, these books assume one year of high school algebra.

Intermediate Algebra. By J. R. Britton and L. C. Snively (University of Colorado). Rinehart and Company, 1947. 337 pages, $3.00. Review: vol. 55, p. 376 (1948).

This volume consists of the first twelve chapters of the authors' survey text plus four additional chapters.

Intermediate Algebra for Colleges. By E. B. Miller (Illinois College). The Ronald Press Company, 1947. 361 pages, $3.00. Review: vol. 55, p. 177 (1948).

Intermediate Algebra. By R. S. Underwood (Texas Technological College), T. R. Nelson (Texas A. and M. College), and S. Selby (University of Akron). The Macmillan Company, 1947. 283 pages, $3.00. Review: vol. 55, p. 517 (1948).

Intermediate Algebra for Colleges. By W. L. Hart (University of Minnesota). D. C. Heath and Company, 1948. 276 pages, $3.50. Review: vol. 56, p. 194 (1949).

Intermediate Algebra for Colleges. By P. R. Rider (Washington University). The Macmillan Company, 1949. 242 pages, $3.00. Review: vol. 57, p. 494 (1950).

Intermediate College Algebra. By E. M. J. Pease. Prentice-Hall, Inc., 1950. 420 pages, $3.50. Review: vol. 59, p. 192 (1952).

Elements of Algebra. By L. C. Peck (Ohio University). McGraw-Hill Book Company, 1950. 230 pages, $3.25. Review: vol. 59, p. 192 (1952).

This text is written primarily for college students who have had no algebra in high school, and thus presupposes no mathematics background other than ordinary arithmetic.

Intermediate Algebra. By R. W. Brink (University of Minnesota). Appleton-Century-Crofts, Inc., 1951. 295 pages, $3.25.

This book, a second edition, consists of the first fifteen chapters of Brink's College Algebra, Second Edition.

Algebra for Commerce and Liberal Arts. By A. K. Bettinger (Creighton University) and W. A. Dwyer. Pitman Publishing Company, 1951. 225 pages, $3.00. Review: vol. 59, p. 192 (1952).

Intermediate Algebra. By P. K. Rees (Louisiana State University) and F. W. Sparks (Texas Technological College). McGraw-Hill Book Company, 1951. 328 pages, $3.50. Review: vol. 59, p. 192 (1952).

Intermediate Algebra for Colleges. By J. B. Rosenbach and E. A. Whitman (Carnegie Institute of Technology). Ginn and Company, 1951. 241 pages, $3.00. Review: vol. 61, p. 274 (1954).

This book, designed for those students who have had a minimum of algebra, covers those topics usually completed in two years of high school algebra through progressions, but written at a college maturity level.

Beginning Algebra for College Students. By L. L. Lowenstein (Kent State University). John Wiley and Sons, Inc., 1953. 279 pages, $3.50.

Geared for an introductory or terminal course, this introduces algebra through the use of arithmetic.

Intermediate Algebra for College Students (Revised edition). By T. S. Peterson, (Portland State Extension Center). Harper and Brothers, 1953. 369 pages, $3.00.

Elements of Algebra. By V. B. Caris (Ohio State University). Ginn and Company, 1953. 307 pages, $3.30. Review: vol. 61, p. 272 (1954).

Intermediate Algebra (Revised edition). By L. J. Adams (Santa Monica City College). Henry Holt and Company, 1954. 366 pages, $3.40.

Algebra for College Students. By W. M. Whyburn (University of North Carolina) and P. H. Daus (University of California at Los Angeles). Prentice-Hall, Inc., 1955. 255 pages, $4.25.

COLLEGE ALGEBRA

College Algebra. By F. M. Morgan (Clark School). American Book Company, 1943. 374 pages, $3.00. Review: vol. 51, p. 41 (1944).

Brief College Algebra. By W. L. Hart (University of Minnesota). D. C. Heath and Company, 1947. 292 pages, $3.25.

College Algebra. By E. R. Heineman (Texas Technological College). The Macmillan Company, 1947. 359 pages, $3.50. Review: vol. 55, p. 437 (1948).

College Algebra. By F. S. Nowlan (University of British Columbia). McGraw-Hill Book Company, 1947. 371 pages, $4.00. Review: vol. 55, p. 374 (1948).

College Algebra. By T. S. Peterson (University of Oregon). Harper and Brothers, 1947. 334 pages, $3.00. Review: vol. 56, p. 49 (1949).

College Algebra. By Moses Richardson (Brooklyn College). Prentice-Hall, Inc., 1947. 472 pages, $3.95. Review: vol. 55, p. 107 (1948).

College Algebra. By Gordon Fuller (Texas Technological College). D. Van Nostrand Company, Inc., 1948. 255 pages, $3.25. Review: vol. 56, p. 281 (1949).

This text is for freshman college students who are to take a course in algebra composed of material of high school algebra and some chapters of a more advanced nature, usually classed as college algebra.

College Algebra. By L. M. Reagan (University of Wichita), E. R. Ott (Rutgers University), and D. T. Sigley (Johns Hopkins University). Rinehart and Company, 1948. 447 pages, $4.00.

This book departs from conventional college algebra texts in the order of topics and in the use of the inductive approach in developing new concepts.

College Algebra. By H. A. Simmons (Northwestern University). The Macmillan Company, 1948. 619 pages, $4.00. Review: vol. 56, p. 193 (1949).

College Algebra. By E. A. Cameron and E. T. Browne (both of University of North Carolina). Henry Holt and Company, 1949. 406 pages, $3.80. Review: vol. 57, p. 50 (1950).

College Algebra (Third edition). By J. B. Rosenbach and E. A. Whitman (both of Carnegie Institute of Technology). Ginn and Company, 1949. 565 pages, $3.50. Review: First edition, vol. 41, p. 258 (1934); Revised edition, vol. 56, p. 647 (1949).

The flexible organization of the third edition makes it adjustable to the

needs and abilities of different groups of students and to courses of various lengths and objectives.

College Algebra. By H. A. Bender (University of Rhode Island). Pitman Publishing Company, 1950. 452 pages, $3.75.

College Algebra. By E. B. Miller (Illinois College) and R. M. Thrall (University of Michigan). The Ronald Press, 1950. 493 pages, $4.25. Review: vol. 58, p. 200 (1951).

A textbook for first-year college students who propose to make a career of mathematics or of some science where a thorough knowledge of mathematics is indispensable.

Essentials of College Algebra. By J. B. Rosenbach and E. A. Whitman (Carnegie Institute of Technology). Ginn and Company, 1950. 352 pages, $3.00. Review: vol. 59, p. 192 (1952).

This textbook is based on, but shorter than, the authors' *College Algebra.*

College Algebra. By R. W. Brink (University of Minnesota). Appleton-Century-Crofts, Inc., 1951. 495 pages, $4.00.

This book, a second edition, is designed for college students who are in need of a review of high-school higher algebra.

Algebra: College Course. By R. W. Brink (University of Minnesota). Appleton-Century-Crofts, Inc., 1951. 378 pages, $3.50.

This book is intended for college students who have had adequate instruction in high-school algebra and do not require a review of the subject.

College Algebra. By H. K. Fulmer and Walter Reynolds (Georgia Institute of Technology). Ginn and Company, 1951. 218 pages, $3.00. Review: vol. 59, p. 192 (1952)

This short (quarter or semester) course in algebra condenses, into as brief a text as possible, a review of the essentials of elementary algebra together with an adequate treatment of the theory of equations.

College Algebra. By H. L. Rietz, A. R. Crathorne, and J. W. Peters (University of Illinois). Henry Holt and Company, 1951. 387 pages, $3.50. Review: vol. 59, p. 192 (1952).

This contains new material on functional notation, graphs, and elementary algebra.

College Algebra. By R. R. Middlemiss (Washington University). McGraw-Hill Book Company, 1952. 344 pages, $3.50. Review: vol. 61, p. 273 (1954).

College Algebra. By R. H. Bardell and Abraham Spitzbart (University of Wisconsin, Milwaukee Extension). Addison-Wesley Publishing Company, Inc., 1953. 208 pages, $3.50. Review: vol. 61, p. 270 (1954).

College Algebra. By J. R. Britton and L. C. Snively (University of Colorado). Rinehart and Company, 1953. 502 pages, $5.00. Review: vol. 61, p. 271 (1954).

College Algebra (Fourth edition). By W. L. Hart (University of Minnesota). D. C. Heath and Company, 1953. 420 pages, $3.50.

Algebra for College Students. By R. R. Middlemiss (Washington University). McGraw-Hill Book Company, 1953. 394 pages, $3.75. Review: vol. 61, p. 274 (1954).

This text is identical with the author's *College Algebra* with the following exceptions: the first three chapters of *College Algebra* have been replaced by five chapters, covering this same elementary material at a more leisurely pace.

College Algebra. By H. G. Apostle (Grinnell College). Henry Holt and Company, 1954. 422 pages, $4.50. Review: vol. 62, p. 192 (1955).

Algebra for College Students. By J. R. Britton and L. C. Snively (University of Colorado). Rinehart and Company, 1954. 537 pages, $4.25. Review: vol. 62, p. 192 (1955).

College Algebra. By M. W. Keller (Purdue University). Houghton Mifflin Company, 1954. 471 pages, $4.90.

College Algebra (Third edition). By P. K. Rees (Louisiana State University) and F. W. Sparks (Texas Technological College). McGraw-Hill Book Company, 1954. 414 pages, $4.25. Review: vol. 62, p. 192 (1955).

Understanding College Algebra. By Samuel Selby (University of Akron), E. R. Smith (Iowa State College) and Murray Kleiman (City College of New York). The Dryden Press, 1954. 505 pages, $3.50. Review: vol. 62, p. 192 (1955).

College Algebra. By P. R. Rider (Washington University). The Macmillan Company, 1955 (in press).

This is a revised edition of the author's *College Algebra*, 1940.

TRIGONOMETRY

Plane and Spherical Trigonometry. By H. A. Simmons (Northwestern University). John Wiley and Sons, Inc., 1945. 387 pages, $3.75 with tables, $3.00 without tables. Review: vol. 53, p. 95 (1946).

A revision of Simmons and Gore's *Plane Trigonometry*. Gives applications to surveying, navigation and nautical astronomy, with appendixes on complex numbers and the slide rule.

Essentials of Plane and Spherical Trigonometry (Revised edition). By Clifford Bell (University of California at Los Angeles) and T. Y. Thomas (Indiana

University). Henry Holt and Company, 1946. 246 pages, $3.15 with tables, $2.60 without tables.

Trigonometry. By H. K. Hughes and G. T. Miller (both at Purdue University). John Wiley and Sons, Inc., 1946. 175 pages, $3.25 with tables, $2.75 without tables.

Plane Trigonometry. By W. K. Morrill (Johns Hopkins University). Rinehart and Company, 1946. 245 pages. $2.50.

Plane Trigonometry. By E. B. Mode (Boston University). Prentice-Hall, Inc., 1947. 214 pages, $2.95. Review: vol. 55, p. 176 (1948).

Plane and Spherical Trigonometry. By J. Shibli (Pennsylvania State College). Ginn and Company, 1949. 262 pages, $3.60. Review: vol. 58, p. 645 (1951).

This book contains practical applications including problems in surveying, engineering, physics, navigation, aviation, and astronomy.

Plane Trigonometry. By J. J. Corliss (University of Illinois) and W. V. Berglund (University of Illinois). Houghton Mifflin Company, 1950. 388 pages, $3.25. Review: vol. 58, p. 645 (1951).

Plane Trigonometry. By H. M. Dadourian (Trinity College). Addison-Wesley Publishing Company, Inc., 1950. 194 pages, $3.50.

This is a textbook for a brief course in trigonometry intended particularly for science and engineering students. Of special interest is the author's application of trigonometry to other topics, including an entire chapter devoted to applications to physics.

Plane Trigonometry. By Gordon Fuller (Alabama Polytechnic Institute). McGraw-Hill Book Company, 1950. 202 pages, $3.25 with tables, $2.75 without tables. Review: vol. 58, p. 645 (1951).

Plane Trigonometry. By E. R. Heineman (Texas Technological College). McGraw-Hill Book Company, 1950. 252 pages, $3.25 with tables, $2.75 without tables. Review: vol. 58, p. 645 (1951).

This is an alternate edition of the author's *Plane Trigonometry*, published in 1942, containing a new set of problems and exercises.

Plane and Spherical Trigonometry (Revised edition). By J. A. Northcott (Columbia University). Rinehart and Company, 1950. 234 pages, $3.50 with tables, $2.50 without tables.

Plane and Spherical Trigonometry (Fifth edition). By C. I. Palmer, C. W. Leigh, and Spofford Kimball (University of Maine). McGraw-Hill Book Company, 1950. 164 pages, $4.50 with tables, $3.25 without tables.

Plane and Spherical Trigonometry. By Moses Richardson (Brooklyn College). The Macmillan Company, 1950. 343 pages, $3.75 with tables, $3.40 without tables. Review: vol. 60, p. 720 (1953).

This text presents a full treatment of plane and spherical trigonometry adaptable to long or short courses with varying emphasis.

Plane and Spherical Trigonometry (Third edition). By H. L. Rietz, J. F. Reilly, and Roscoe Woods (University of Iowa). The Macmillan Company, 1950. 205 pages, $3.00 with tables, $2.75 without tables.

Essentials of Plane Trigonometry. By J. B. Rosenbach, E. A. Whitman and David Moskovitz (all of the Carnegie Institute of Technology). Ginn and Company, 1950. 183 pages, $3.00 with tables, $2.85 without tables. Review: vol. 58, p. 645 (1951).

Trigonometry. By C. T. Holmes (Bowdoin College). McGraw-Hill Book Company, 1951. 153 pages, $3.25 with tables, $2.75 without tables.

This text connects trigonometry more closely with other college mathematics. This is accomplished by introducing some very elementary concepts from analytic geometry and using them consistently in proofs throughout the book.

Plane and Spherical Trigonometry. By L. M. Kells, W. F. Kern, and J. R. Bland (all of the United States Naval Academy). McGraw-Hill Book Company, 1951. 290 pages, $4.00 with tables, $3.25 without tables.

Plane Trigonometry. By L. M. Kells, W. F. Kern, and J. R. Bland (all of the United States Naval Academy). McGraw-Hill Book Company, 1951. 220 pages, $3.75 with tables, $3.25 without tables. Review: vol. 60, p. 720 (1953).

This textbook consists of those sections of the authors' *Plane and Spherical Trigonometry* which are restricted to plane trigonometry.

Plane and Spherical Trigonometry. By F. M. Morgan (Clark School). American Book Company, 1951. 324 pages, $3.25 with tables.

Brief Trigonometry. By E. A. Cameron (University of North Carolina). Henry Holt and Company, 1952. 153 pages, $2.45.

The material is presented so that it can be covered in fewer then thirty assignments.

Trigonometry, Plane and Spherical. By L. L. Smail (Lehigh University). McGraw-Hill Book Company, 1952. 324 pages, $4.00 with tables, $3.25 without tables. Review: vol. 60, p. 720 (1953).

Plane Trigonometry. By F. W. Sparks (Texas Technological College) and P. K. Rees (Louisiana State University). Prentice-Hall, Inc., 1952. 199 pages, $3.25.

Plane Trigonometry. By D. H. Ballou (Middlebury College), and F. H. Steen (Allegheny College). Ginn and Company, 1953. 160 pages, $3.25 with tables.

Plane and Spherical Trigonometry. By D. H. Ballou (Middlebury College), and F. H. Steen (Allegheny College). Ginn and Company, 1953. 218 pages, $3.50 with tables, $3.25 without tables.

This textbook includes the authors' *Plane Trigonometry* as well as additional material and is constructed for a maximum of self-study and self-drill.

Trigonometry. By J. F. Randolph (University of Rochester). The Macmillan Company, 1953. 220 pages, $3.00. Review: vol. 61, p. 656 (1954).

The body of this text is devoted to purely trigonometric concepts and their applications, but pertinent principles of analytic geometry and logarithms are included in appendices. A bridge between high school and college work is provided by another appendix containing a review of elementary algebra.

Plane Trigonometry. By P. R. Rider (Washington University). The Macmillan Company, 1953. 180 pages. $3.00. Review: vol. 61, p. 656 (1954).

This book consists of the plane trigonometry sections of the author's *First-Year Mathematics for Colleges.*

Plane Trigonometry. By A. W. Weeks and H. G. Funkhouser (Phillips Exeter Academy). D. Van Nostrand Company, Inc., 1953. 193 pages, $2.68. Review: vol. 61, p. 656 (1954).

Trigonometry. By W. L. Hart (University of Minnesota). D. C. Heath and Company, 1954. 368 pages, $3.75.

Trigonometry. By E. P. Vance (Oberlin College). Addison-Wesley Publishing Company, 1954. 158 pages, $3.00. Review: to appear in June–July (1955).

This textbook emphasizes analytic trigonometry rather than triangle solution, although both are included. The rectangular coordinate system is introduced immediately and used throughout the book.

Trigonometry. By Roy Dubisch (Fresno State College). The Ronald Press Company, 1955. 396 pages, $5.00. Review: to appear in June–July (1955).

This textbook defines the circular functions as functions of a real variable, by use of the unit circle. It presents a comprehensive treatment.

COLLEGE ALGEBRA AND TRIGONOMETRY

College Algebra and Trigonometry. By F. H. Miller (The Cooper Union School of Engineering). John Wiley and Sons, Inc., 1945. 324 pages, $4.00. Review: vol. 52, p. 450 (1945).

This textbook for a basic integrated course is an attempt to give an integrated treatment of the two subjects on the college level and to lay a suitable

foundation for the study of more advanced mathematics. Throughout, algebra and trigonometry support and complement each other, and the techniques of both are applied continually.

College Algebra and Plane Trigonometry. By J. H. Zant, (Oklahoma Agricultural and Mechanical College). Ginn and Company, 1953. 387 pages, $4.00.

Throughout this book, which gives the student an integrated foundation in algebra and plane trigonometry, principles and skills from both fields are applied continually. The natural relations between them are brought out but the material is not combined when the integration would be artificial.

College Algebra and Plane Trigonometry. By R. H. Bardell and Abraham Spitzbart (both of University of Wisconsin, Milwaukee Extension). Addison-Wesley Publishing Company, Inc., March 1955.

This textbook which integrates Algebra and Plane Trigonometry, has avoided artificial integration for integration's sake. It is designed for either a one or a two semester course. The algebra portion is similar to that of the authors' *College Algebra.*

Unified Algebra and Trigonometry. By E. P. Vance (Oberlin College). Addison-Wesley Publishing Company, Inc., March, 1955.

This is a completely integrated textbook which emphasizes analytic trigonometry rather than triangle solution and has many illustrative examples from analytic geometry. It is designed primarily for students who plan to continue in mathematics or science and may be used for a one or two semester course.

ANALYTIC GEOMETRY

New Analytic Geometry. By P. F. Smith, A. S. Gale (University of Rochester), and J. H. Neelley (Carnegie Institute of Technology). Ginn and Company, 1945. 326 pages, $3.20.

Analytic Geometry. By R. R. Middlemiss (Washington University). McGraw-Hill Book Company, 1945. 306 pages, $3.75. Review: vol. 53, p. 214 (1946).

Analytic Geometry. By F. D. Murnaghan (Johns Hopkins University). Prentice-Hall, Inc., 1946. 402 pages, $3.75. Review: vol. 53, p. 530 (1946).

Analytic Geometry. By F. S. Nowlan (University of British Columbia). McGraw-Hill Book Company, 1946. 355 pages, $3.75. Review: vol. 53, p. 529 (1946).

Essentials of Analytic Geometry. By D. R. Curtiss and E. J. Moulton (Northwestern University). D. C. Heath and Company, 1947. 269 pages, $3.25. Review: vol. 56, p. 48 (1949).

Brief Analytic Geometry. By T. E. Mason and C. T. Hazard (Purdue University). Ginn and Company, 1947. 228 pages, $3.00.

This textbook is a revision of the authors' longer *Analytic Geometry.*

Analytic Geometry. By D. S. Nathan (College of the City of New York) and Olaf Helmer (Douglas Aircraft Co.). Prentice-Hall, Inc., 1947. 402 pages, $3.75. Review: vol. 55, p. 595 (1948).

Analytic Geometry. By P. R. Rider (Washington University). The Macmillan Company, 1947. 383 pages, $3.50. Review: vol. 56, p. 487 (1949).

Analytic Geometry. By R. S. Underwood (Texas Technological College) and F. W. Sparks (Texas Technological College). Houghton Mifflin Company, 1947. 255 pages, $3.25. Review: vol. 55, p. 597 (1948).

Analytic Geometry (Fourth edition). By C. E. Love (University of Michigan). The Macmillan Company, 1948. 306 pages, $3.50.

Analytic Geometry. By P. K. Rees (Louisiana State University) and E. D. Mouzon (Southern Methodist University). The Dryden Press, 1948. 305 pages, $2.75. Review: vol. 56, p. 488 (1949).

Analytic Geometry. By Roscoe Woods (University of Iowa). The Macmillan Company, 1948. 322 pages, $3.50.

A second edition, this book is designed for either a short course (first ten chapters) or a longer course. Its special feature is the representation of the quadric surfaces by means of photographs and wire models.

Analytic Geometry. By J. J. Corliss, I. K. Feinstein, and H. S. Levin (University of Illinois). Harper and Brothers, 1949. 370 pages, $3.25.

This text considers both plane and solid analytic geometry.

Analytic Geometry. By A. L. Nelson, K. W. Folley, and W. M. Borgman (Wayne University). The Ronald Press Company, 1949. 215 pages, $3.25. Review: vol. 57, p. 426 (1950).

Analytic Geometry. By L. M. Kells and H. C. Stotz (both of U. S. Naval Academy). Prentice-Hall, Inc., 1949. 288 pages, $3.50. Review: vol. 58, p. 201 (1951).

Analytic Geometry. By Robin Robinson (Dartmouth College). McGraw-Hill Book Company, 1949. 147 pages, $3.25. Review: vol. 57, p. 124 (1950).

Analytic Geometry (Revised edition). By C. H. Sisam (Colorado College). Henry Holt and Company, 1949. 304 pages, $3.40.

Analytic Geometry (Third edition). By W. A. Wilson and J. I. Tracey (Yale University). D. C. Heath and Company, 1949. 328 pages, $3.50.

Analytic Geometry. By R. D. Douglass and S. D. Zeldin (Massachusetts Institute of Technology). McGraw-Hill Book Company, 1950. 212 pages. $3.75.

Elements of Analytic Geometry. By W. L. Hart, (University of Minnesota). D. C. Heath and Company, 1950. 229 pages, $3.25. Review: vol. 58, p. 275 (1951).

Elements of Analytic Geometry. By C. E. Love (University of Michigan). The Macmillan Company, 1950. 218 pages, $3.00.

This text, similar to the author's *Analytic Geometry*, has been written in a similar fashion but designed for a briefer course.

Analytic Geometry. By J. W. Cell (North Carolina State College). John Wiley and Sons, Inc., 1951. 326 pages, $3.75.

This book contains numerous illustrations and problems in science and engineering.

Brief Course in Analytics. By M. A. Hill, Jr., and L. J. Burton (both of the University of North Carolina). Henry Holt and Company, 1951. 224 pages, $2.75.

Analytic Geometry. By Gordon Fuller (Texas Technological College). Addison-Wesley Publishing Company, Inc., 1954. 216 pages, $3.85. Review: vol. 62, p. 196 (1955).

An important feature is the introduction of vectors and their applications.

Analytic Geometry (Second edition). By E. S. Smith, Meyer Salkover, and H. K. Justice (University of Cincinnati). John Wiley and Sons, Inc., 1954. 306 pages, $4.00. Review: vol. 62, p. 196 (1955).

Analytic Geometry (Third edition). By F. H. Steen (Allegheny College) and D. H. Ballou (Middlebury College). Ginn and Company, 1955.

UNIFIED FRESHMAN TEXTS

An Introduction to Mathematical Analysis (Revised edition). By F. L. Griffin (Reed College). Houghton Mifflin Company, 1936. 546 pages, $4.50. Review: vol. 44, p. 172 (1937).

This is a thoroughly integrated treatment of calculus, analytic geometry, algebra, and trigonometry.

Introductory College Mathematics. By W. E. Milne (Oregon State College) and D. R. Davis (Montclair State Teachers College). Ginn and Company, 1941. 438 pages, $4.50.

This textbook correlates the essential topics of college algebra, trigonometry, analytic geometry, and some principles of the calculus. The early introduction of the simpler elements of the calculus is one of the methods of the correlation. An elementary treatment of finite differences and curve fitting is included.

Fundamentals of Mathematics. By Moses Richardson (Brooklyn College). The Macmillan Company, 1941. 525 pages, $5.00. Review: vol. 48, p. 472 (1941).

This textbook covers, in addition to the traditional subject matter, many topics which emphasize the broad scope of mathematical ideas. It begins with a logical treatment of the number system and presents other topics in this same manner. In addition, there are discussions of some of the simpler important problems of pure mathematics, many of them of the non-traditional type.

College Mathematics. By C. H. Sisam (Colorado College). Henry Holt and Company, 1946. 469 pages, $4.50. Review: vol. 54, p. 118 (1947).

This is a general introduction of unified college algebra, trigonometry, and analytic geometry.

Elementary Concepts of Mathematics. By B. W. Jones (University of Colorado). The Macmillan Company, 1947. 294 pages, $4.75. Review: vol. 55, p. 515 (1948).

This text is designed for students who have had a minimum of training in mathematics, who do not expect to specialize in mathematics or science, but who want a firmer grounding in what useful mathematics they have had, and such additional training as they may find interesting and useful in later life.

First Year College Mathematics with Applications. By P. H. Daus (University of California at Los Angeles) and W. M. Whyburn (University of North Carolina). The Macmillan Company, 1949. 495 pages, $5.00. Review: vol. 57, p. 124 (1950).

This book provides an ample background for the study of the calculus, and simultaneously, integrates the subjects of college algebra, analytic geometry, and analytic trigonometry. Applications taken from science and engineering illustrate each of the principles discussed.

Freshman Mathematics. By H. L. Slobin and W. E. Wilbur, completely rewritten by C. V. Newsom (Associate Commissioner of Education, The State of New York). Rinehart and Company, 1949. 559 pages, $5.25. Review: vol. 57, p. 428 (1950).

Algebra, trigonometry, and analytical geometry are presented together as a tandem course and there is no repetition or overlapping of material.

First Year Mathematics for Colleges. By P. R. Rider (Washington University). The Macmillan Company, 1949. 714 pages, $5.00. Review: vol. 58, p. 53 (1951).

The topics are logically arranged and grouped about the function concept, yet the individual chapters are sufficiently independent for the teacher to adapt the book to his own sequence and needs.

Introduction to Mathematics. By H. R. Cooley, David Gans, Morris Kline and H. E. Wahlert (all of New York University). Houghton Mifflin Company, 1949. 710 pages, $5.00.

This book is intended for liberal arts students and presents a great variety of topics with a cultural rather than a technical approach.

Living Mathematics. By R. S. Underwood and F. W. Sparks (Texas Technological College). McGraw-Hill Book Company, 1949. 363 pages, $5.50.

This textbook is an elementary introduction up to but not including calculus. Part I contains the conventional material for a one-semester course in College Algebra. Part II provides a terminating course, as well as much interesting reference material.

Basic Mathematical Analysis. By H. G. Ayre (Western Illinois State College). McGraw-Hill Book Company, 1950. 584 pages, $6.00.

This textbook brings together coherently the essentials of algebra, trigonometry, analytic geometry and a minimum of the easier concepts of the calculus.

A First Course in Mathematics. By Pompey Mainardi, Carl Konove and Edward Baker (Newark College of Engineering). D. Van Nostrand Company, Inc., 1950. 380 pages, $5.85.

This third edition is designed for freshman students of engineering and the physical sciences with emphasis on the technique of applying mathematics to physical situations. It includes the usual material including three dimensional geometry with a chapter on derivatives and integrals.

Primer of College Mathematics. By J. F. Randolph (University of Rochester). The Macmillan Company, 1950. 545 pages, $5.00. Review: vol. 58, p. 350 (1951).

This text is a unification of college algebra, trigonometry, and analytic geometry with an introduction to calculus. Sufficient material is included to permit considerable selection even in a three-semester course. Even though the subjects are integrated, the book may be used in a fairly traditional presentation of the three elementary courses.

Elements of Mathematical Analysis. By S. E. Urner (Los Angeles City College) and W. B. Orange. Ginn and Company, 1950. 562 pages, $4.50. Review: vol. 58, p. 577 (1951).

In this textbook the simpler elements of the calculus are introduced at an early stage. Algebraic techniques are included, trigonometric material is complete with the main emphasis on the analytic side. Supplementary material includes discussions of such fundamental topics as limits, continuity, existence of a derivative, and infinity.

College Mathematics (Second edition). By W. W. Elliott (Duke University) and E. R. C. Miles (Johns Hopkins University). Prentice-Hall, Inc., 1951. 472 pages, $4.95.

This text is a non-integrated treatment of algebra, trigonometry, analytic geometry, and the elements of calculus.

General College Mathematics. By W. L. Ayres, C. G. Fry, and H. F. S. Jonah (Purdue University). McGraw-Hill Book Company, 1952. 280 pages, $3.75. Review: vol. 60, p. 486 (1953).

This textbook is a relatively simple book for freshman mathematics designed as a terminal course for scientific students. Important operations in arithmetic are reviewed, leading the student through the elements of algebra, trigonometry, and the elementary concepts of analytic geometry. The book concludes with a chapter on simple logic.

Basic Skills in Mathematics. By H. V. Price and L. A. Knowler (both of the State University of Iowa). Ginn and Company, 1952. 249 pages, $3.25.

This textbook is designed to provide basic training in mathematics to meet the demands of various programs and is planned particularly for students whose background is inadequate, and whose interests lie in other fields. The material covered is almost identical to that of the Commission on Post-war Plans of the National Council of Teachers of Mathematics.

Elementary Analysis. By K. O. May (Carleton College). John Wiley and Sons, Inc., 1952. 625 pages, $5.00. Review: vol. 59, p. 708 (1952).

Although the subject matter is traditional, and includes an introduction to calculus, the approach is quite modern with much time spent on basic ideas. Carefully drawn explanations of involved mathematical concepts are given and the book presents more material, more problems, and more functional information than is usually found in books of this type.

Fundamentals of College Mathematics. By R. E. Johnson (Smith College), N. H. McCoy (Smith College) and Anne F. O'Neill (Wheaton College). Rinehart and Company, 1953. 479 pages, $6.00. Review: vol. 61, p. 58 (1954).

This basic textbook for integrated courses in trigonometry, analytic geometry, and calculus, is designed primarily for liberal arts freshmen with at least three units of entrance mathematics. Emphasis is on the understanding of basic ideas rather than on the development of problem-solving techniques.

A First Year of College Mathematics. By R. W. Brink (University of Minnesota). Appleton-Century-Crofts, Inc., 1954. 725 pages, $5.00.

This standard textbook has been reexamined and rewritten and presents a

complete and integrated course in the material of college algebra, trigonometry, and analytic geometry.

Fundamentals of College Mathematics. By J. C. Brixey and R. V. Andree (both of the University of Oklahoma). Henry Holt and Company, 1954. 620 pages, $5.90. Review: vol. 62, p. 193 (1955).

This textbook for college freshmen includes college algebra, trigonometry, and analytic geometry (which are carefully integrated) as well as an introduction to differential and integral calculus and to mathematical statistics.

Introductory College Mathematics. By C. G. Jaeger (Pomona College) and H. M. Bacon (Stanford University). Harper and Brothers, 1954. 377 pages, $4.75.

This textbook is designed for use on the elementary level, and provides a unified treatment of basic topics in algebra, trigonometry, and analytic geometry together with an introduction to the calculus. The unifying element throughout the book is the function concept. It is intended for use in a full year course, and presupposes one year each of high school algebra and geometry.

Introductory College Mathematics. By Adele Leonhardy (Stephens College). John Wiley and Sons, Inc., 1954. 459 pages, $4.90.

The material in this text is organized around the nature of mathematics and its role in modern civilization and is designed to meet the requirements of a broad group of readers: those interested in a terminal work, those whose main field of interest is in education, humanities, or the social sciences, and those who wish a broader treatment than is ordinarily offered in introductory works.

An Introduction to College Mathematics. By C. V. Newsom (Associate Commissioner of Education, State of New York) and Howard Eves (University of Maine). Prentice-Hall, Inc., 1954. 416 pages, $5.75. Review: vol. 61, p. 724 (1954).

This is a revised edition by Eves of Newsom's textbook which is designed for a terminal course in first year college mathematics. The revision has introduced certain historical material and incorporated a chapter which introduces the idea of calculus.

Principles of Mathematics. By C. B. Allendoerfer (University of Washington), C. O. Oakley (Haverford College). McGraw-Hill Book Company, 1955. 450 pages, in press.

Basic Mathematics for General Education (Second edition). By H. C. Trimble and L. C. Peck (both of Iowa State Teachers College) and F. C. Bolser. Prentice-Hall, Inc., 1955. 336 pages, in press.

This book is an elementary introduction to mathematics for the general student. It emphasizes ideas rather than techniques.

ANALYTIC GEOMETRY AND CALCULUS

Analytic Geometry and Calculus. By F. S. Woods and F. H. Bailey. Ginn and Company, 1938. 524 pages, $5.00.

This book opens with the analytic geometry of two dimensions. The geometry of three dimensions is treated later when it is required for the study of functions of two variables. The calculus is introduced early through the discussion of slope and area and from this point on is intermingled with analytic geometry.

Analytic Geometry and Calculus. By H. B. Phillips (The Massachusetts Institute of Technology). John Wiley and Sons, Inc., 1946. 504 pages, $5.50. Review: vol. 54, p. 56 (1947).

This textbook treats analytic geometry and calculus in the form and order in which they are required for courses in science and engineering. It covers the essentials but does not burden the student with a mass of detail.

Analytic Geometry and Calculus. By J. F. Randolph (University of Rochester) and Mark Kac (Cornell University). The Macmillan Company, 1946. 642 pages, $5.25. Review: vol. 54, p. 293 (1947).

In this textbook analytic geometry and calculus are treated together in such a way that each complements the other. Strong emphasis is placed on the functional notation. Integration is introduced early and in the first chapters a review is given of some of the fundamental algebraic notions, such as inequalities and absolute values.

Introduction to Analytic Geometry and the Calculus. By H. M. Dadourian (Trinity College). The Ronald Press Company, 1949. 246 pages, $3.50. Review: vol. 57, p. 354 (1950).

Avoiding the formal manner of presentation, this textbook is aimed at presenting the elements of analytic geometry and the calculus in order to make these subjects intelligible to the beginner. New concepts are introduced through an appeal to the student's experience. Clear and concise directions are given in solving problems.

Analytic Geometry and Calculus: A Unified Treatment. By F. H. Miller (The Cooper Union School of Engineering). John Wiley and Sons, Inc., 1949. 658 pages, $5.50.

This textbook is a correlated study of analytic geometry and calculus designed for engineering or science students who desire a unified treatment entailing the early introduction of differential and integral calculus.

Analytical Geometry and Calculus. By H. J. Gay. Edited by R. K. Morley (Worcester Polytechnic Institute). McGraw-Hill Book Company, 1950. 524 pages, $5.00. Review: vol. 58, p. 577 (1951).

This textbook is intended to cover the elements of analytical geometry and calculus in an order of topics that will introduce the basic ideas of calculus fairly early, but not until the student has had enough work with coordinates to make the graphs of the common curves familiar, so that they may be used readily for illustrative purposes in calculus.

Calculus and Analytic Geometry. By C. T. Holmes (Bowdoin College). McGraw-Hill Book Company, 1950. 416 pages, $5.00. Review: vol. 58, p. 433 (1951).

This textbook is planned for a combination course in which the concepts and techniques of the calculus are the main objectives. The analytic geometry material is closely interwoven with calculus throughout the first half of the book, and the concept of integration is introduced early. The text is suitable for either arts and science students or engineering students.

Analytic Geometry and Calculus. By L. M. Kells (United States Naval Academy). Prentice-Hall, Inc., 1950. 608 pages, $5.50. Review: vol. 58, p. 577 (1951).

This textbook unifies the analytic geometry and calculus and covers all the ordinary material.

Analytic Geometry and Calculus. By W. R. Longley, P. F. Smith and W. A. Wilson (all of Yale University). Ginn and Company, 1951. 598 pages, $5.00.

This textbook combines analytic geometry and calculus including one chapter on differential equations, so that a student may be introduced to the calculus before completing analytic geometry.

Calculus and Analytic Geometry. By G. B. Thomas, Jr. (Massachusetts Institute of Technology). Addison-Wesley Publishing Company, Inc., 1953. 731 pages, $7.50. Review: vol. 61, p. 435 (1954).

In this textbook on analytic geometry and calculus, designed primarily for students of science and engineering, integration is introduced early, with many applications to geometry and mechanics. In addition to standard topics, this book contains discussions of determinants and hyperbolic functions, introduction to vector analysis, infinite series (including Taylor and Fourier Series) and the theory of complex variables, including the Cauchy-Riemann differential equations.

Analytic Geometry and Calculus. By L. L. Smail (Lehigh University). Appleton-Century-Crofts, Inc., 1953. 714 pages, $5.50. Review: vol. 61, p. 435 (1954).

After the basic ideas of analytic geometry appear, the fundamental concepts and processes of calculus are introduced. After considerable discussion on differentiation with simple applications, the indefinite integral and the definite integral with simple applications, the conic sections are introduced. More complicated functions follow, and all ordinary topics such as solid analytic geometry,

partial derivatives and multiple integrals and series are considered. The last chapter deals with elementary differential equations.

Introductory Calculus with Analytic Geometry. By E. G. Begle (Yale University). Henry Holt and Company, 1954. 301 pages, $4.50. Review: vol. 62, p. 54 (1955).

This textbook, in a completely rigorous fashion, covers the differentiation and integration of algebraic, logarithmic, exponential, and trigonometric functions, and includes the elements of plane analytic geometry through conics. It includes much more theory than most calculus textbooks contain.

CALCULUS

Elements of Calculus. By W. A. Granville (Gettysburg College), P. F. Smith, and W. R. Longley (both of Yale University). Ginn and Company, 1946. 549 pages, $4.75. Review: vol. 53, p. 332 (1946).

This extremely well-known calculus is arranged so that the applications of the differential and integral calculus can be presented early in the course.

Differential and Integral Calculus (Second edition). By R. R. Middlemiss (Washington University). McGraw-Hill Book Company, 1946. 497 pages, $4.75.

Calculus. By F. H. Miller (The Cooper Union School of Engineering). John Wiley and Sons, Inc., 1946. 416 pages, $4.50. Review: vol. 54, p. 176 (1947).

Calculus (Revised edition). By A. L. Nelson, K. W. Folley and W. M. Borgman (all of Wayne University). D. C. Heath and Company, 1946. 386 pages, $4.00.

Unified Calculus. By E. S. Smith, Meyer Salkover and H. K. Justice (all of the University of Cincinnati). John Wiley and Sons, Inc., 1947. 507 pages, $4.75. Review: vol. 56, p. 562 (1949).

This textbook gives particular attention to the subject as a whole with great emphasis on the practical applications of calculus to both physics and mechanics. Although it is designed primarily for engineering students, it may be used by liberal arts students as well.

Differential and Integral Calculus. By F. M. Morgan (Director of Clark School, Hanover, N. H.). American Book Company, 1948. 400 pages, $3.25.

This textbook is designed for an elementary course and is characterized by its brevity and non-technical approach. The simpler parts of integration are introduced early so as to assist students who are taking science courses simultaneously with the calculus.

Calculus. By L. M. Kells (U. S. Naval Academy). Prentice-Hall, Inc., 1949. 508 pages, $4.75.

Calculus. By L. L. Smail (Lehigh University). Appleton-Century-Crofts, Inc., 1949. 592 pages, $5.00. Review: vol. 57, p. 427 (1950).

The aim of the textbook is to give due emphasis to the meaning of fundamental concepts, as well as to teach the technique of differentiation and integration and the methods of applying this technique. Much attention has been given to the careful formulation of fundamental definitions and theorems. Integration has been introduced early.

Elements of Calculus. By T. S. Peterson (University of Oregon). Harper and Brothers, 1950. 369 pages, $4.50.

Fundamentals of the Calculus. By D. E. Richmond (Williams College). McGraw-Hill Book Company, 1950. 233 pages, $3.75. Review: vol. 58, p. 431 (1951).

This brief textbook, designed for freshman courses, was written to provide sufficient knowledge of the calculus to furnish an adequate background for courses in physics, and to enable the liberal arts student who elects one year of college mathematics to acquire some feeling for mathematical thinking by presenting the material with as much attention to logical clarity as possible. Great emphasis is placed throughout upon fundamental principles, with problems especially designed to contribute to understanding.

Calculus. By Tomlinson Fort (University of Georgia). D. C. Heath and Company, 1951. 576 pages, $5.00. Review: vol. 60, p. 134 (1953).

This textbook, which covers the usual material, differs from others in that it introduces the idea of limit through the use of infinite series. This novel method is introduced in the first chapter, and is used throughout the book.

Calculus. By J. V. McKelvey (Iowa State College). The Macmillan Company, 1951. 405 pages, $4.50. Review: vol. 60, p. 134 (1953).

This textbook, which presents the differential and integral calculus with its geometrical, physical and mechanical applications, emphasizes the rate of change principle in the study of the derivative and stresses the summation definition for the definite integral.

Elements of the Differential and Integral Calculus. By W. A. Granville (Pennsylvania College). Ginn and Company, 1952. 463 pages, $4.00. Review: vol. 18, p. 170 (1911).

This is the 1911 edition of the "original Granville," long the classic college text in calculus, reissued with a foreword by J. W. Lasley, Jr.

Calculus. By J. F. Randolph (University of Rochester). The Macmillan Company, 1952. 483 pages, $6.00. Review: vol. 60, p. 134 (1953).

This is a flexible, well-proportioned treatment of the theory and applications of calculus, and the text maintains a balance between formal problem material

and general concepts. In addition to the standard material, it presents many proofs and several more advanced topics which include Law of Mass Action, Line Integrals, Graphical Integration, and Green's Theorem.

Calculus. By A. H. Sprague (Amherst College). The Ronald Press Company, 1952. 576 pages, $6.50. Review: vol. 60, p. 134 (1953).

This textbook constitutes a logically complete course in the calculus, with the addition of a chapter on polar coordinates and an extensive chapter on solid analytic geometry. The treatment is rigorous, but analytic proofs are accompanied by detailed explanations. Intuitive geometric illustrations pave the way for such proofs.

Differential and Integral Calculus. By Philip Franklin (Massachusetts Institute of Technology). McGraw-Hill Book Company, 1953. 638 pages, $6.00. Review: vol. 62, p. 56 (1955).

This is a complete basic textbook for the full-year course for engineers and science majors. It is so designed that a sequence of material appears for those needing a review of trigonometry and analytic geometry as well as a sequence for those not needing this review.

Calculus. By G. B. Thomas, Jr. (Massachusetts Institute of Technology). Addison-Wesley Publishing Company, Inc., 1953. 614 pages, $6.50. Review: vol. 61, p. 575 (1954).

This textbook for a first course in calculus is designed for students of science and engineering. Integration is introduced at any early stage, and after this every differentiation formula is related to the corresponding integration formula. Thus, students taking a concurrent physics course are able to make early use of this valuable tool.

Calculus. By T. L. Wade (Florida State University). Ginn and Company, 1953. 506 pages, $4.75.

This is a textbook which presents calculus for students with the usual background in trigonometry and analytic geometry. Particular attention is given to the recall of prerequisite mathematics courses as the need arises. The flexibility of arrangement enables the course to be taught with varying degrees of formalism.

Calculus. By C. R. Wylie, Jr. (University of Utah). McGraw-Hill Book Company, 1953. 565 pages, $6.00. Review: vol. 60, p. 722 (1953).

This textbook covers the usual material and is so organized that, by making a judicious choice from among the many worked examples, an instructor can equally well meet the needs and interests of a class of mathematics majors or a class of engineering students.

First Course in Calculus. By H. R. Cooley (Washington Square College, New York University). John Wiley and Sons, Inc., 1954. 643 pages, $6.00. Review: vol. 62, p. 52 (1955).

In this textbook, the emphasis is less on formal proofs than on a clear understanding of the basic concepts. Manipulations are treated thoroughly but only as an efficient means of carrying out the reasoning.

Differential and Integral Calculus. By C. E. Love and E. D. Rainville (both of University of Michigan). The Macmillan Company, 1954. 526 pages. $5.75. Review: to appear in June–July (1955).

This textbook is a complete revision by Rainville of Love's well-known textbook. In addition to the previous material, there is a short appendix containing a rigorous presentation of limits.

Calculus—An Introduction to Analysis and a Tool for the Sciences. By G. M. Merriman (University of Cincinnati). Henry Holt and Company, 1954. 617 pages, $6.50. Review: to appear in June–July (1955).

This textbook includes a careful explanation of the principles and hypotheses of calculus, and proceeds gradually from an intuitive to a rigorous discussion of theorems, hypotheses, and their applications before offering proof.

Calculus (Third edition). By G. F. Sherwood and A. E. Taylor (both at University of California at Los Angeles). Prentice-Hall, Inc., 1954. 624 pages, $5.75. Review: to appear in June–July (1955).

Although many sections have been completely revised, this edition follows the same pattern as the earlier ones. Integration is introduced early and the book is suitable for either arts and science students or engineers.

Differential and Integral Calculus (Second edition). By H. M. Bacon (Stanford University). McGraw-Hill Book Company, 1955. 559 pages, $6.00.

DIFFERENTIAL EQUATIONS

Differential Equations. By R. P. Agnew (Cornell University). McGraw-Hill Book Company, 1942. 341 pages, $4.50. Review: vol. 50, p. 323 (1943).

This textbook may be used, not only for classroom work for good students, but as a reference book from which all teachers may profit.

A Short Course in Elementary Differential Equations. By E. D. Rainville (University of Michigan). The Macmillan Company, 1949. 210 pages, $3.00. Review: vol. 57, p. 694 (1950).

Designed to furnish an introduction to differential equations, this textbook places great emphasis on the development and execution of methods. The omission of infinite series methods has made possible a thorough treatment of topics essential to a first course.

Differential Equations. By H. W. Reddick (New York University). John Wiley and Sons, Inc., 1949. 288 pages, $3.75.

This textbook treats methods of solving ordinary differential equations and problems in applied mathematics involving ordinary differential equations. It includes a chapter on the linear equation of second order, additional material in hyperbolic functions, systems of curves, and vibratory motion.

Elements of Ordinary Differential Equations. By Michael Golomb and M. E. Shanks (both of Purdue University). McGraw-Hill Book Company, 1950. 352 pages, $4.50. Review: vol. 59, p. 50 (1952).

This text is planned for the standard course as it is offered to engineering students, although it also fits many courses for non-engineering students. The subject is presented in such a way as to stimulate the student's imagination and at the time to inculcate correct mathematical thinking. Many techniques not usually found are included.

Differential Equations. By J. E. Powell and C. P. Wells (both of Michigan State College). Ginn and Company, 1950. 205 pages, $3.00. Review: vol. 59, p. 50 (1952).

This textbook is designed for a brief first course in methods of solving elementary differential equations for students majoring in engineering, science, and mathematics. Series solutions are given more than the usual amount of attention.

Differential Equations (Third edition). By H. B. Phillips (The Massachusetts Institute of Technology). John Wiley and Sons, Inc., 1951. 149 pages, $3.00. Review: vol. 59, p. 50 (1952).

An Introduction to the Theory of Differential Equations. By Walter Leighton (Carnegie Institute of Technology). McGraw-Hill Book Company, 1952. 174 pages, $3.75. Review: vol. 60, p. 201 (1953).

An unusually rigorous presentation, designed for pure mathematicians and college students who are majoring in mathematics, this text covers the usual material given in a first course but with exceptional attention to the underlying theory. Emphasis is upon the fundamental existence theorem which is used as a central idea.

Differential Equations. By A. L. Nelson, K. W. Folley, and Max Coral (all of Wayne University). D. C. Heath and Company, 1952. 309 pages, $4.00.

Elementary Differential Equations. By E. D. Rainville (University of Michigan). The Macmillan Company, 1952. 392 pages, $5.00. Review: vol. 60, p. 58 (1953).

This complete introduction to elementary differential equations contains all the material in Rainville's *Short Course*, and in addition, includes such material

as power series methods, and an extensive introduction to Fourier Series and the solution of boundary value problems in partial differential equations.

Differential Equations. By R. C. Yates (United States Military Academy). McGraw-Hill Book Company, 1952. 212 pages, $3.75. Review: vol. 60, p. 202 (1953).

In addition to the usual material, this text includes numerical solutions and solutions by series, an account of Fourier Series, and certain simple boundary value problems.

Differential Equations (Third edition). By Max Morris and O. E. Brown (Case Institute of Technology). Prentice-Hall, Inc., 1952. 361 pages, $4.50.

Elementary Differential Equations. By L. M. Kells (United States Naval Academy). McGraw-Hill Book Company, 1953. 266 pages, $4.00.

Combining theory and applications, this revision deals with the most important types of differential equations. After each type, the author shows interesting and important applications of this type to geometry, electricity, mechanics, physics, and science generally.

Differential Equations with Applications. By Herman Betz, P. B. Burcham, and G. M. Ewing (all of the University of Missouri). Harper and Brothers, 1954. 310 pages, $4.50. Review: vol. 62, p. 130 (1955).

This textbook, in addition to the standard material, includes applications to the physical, biological and engineering sciences not merely as illustrations, but as an integral part of the text. It also includes a thorough treatment of classical operational methods and an introduction to modern operational methods and to non-linear problems.

A First Course in Ordinary Differential Equations. By R. E. Langer (University of Wisconsin). John Wiley and Sons, Inc., 1954. 249 pages, $4.50. Review: vol. 61, p. 437 (1954).

This book is a complete presentation which treats the subject from the standpoint of its mathematical interest and its utility as a technological tool. Emphasis, throughout, is on equations of the first and second order. The book uses an approach which rests solidly upon reason, playing down the necessity for memorization or special tricks.

Differential Equations in Engineering Problems. By M. G. Salvadori and R. J. Schwarz (Columbia University). Prentice-Hall, Inc., 1954. 455 pages, $6.50.

Differential Equations. By Lester Ford (Illinois Institute of Technology). McGraw-Hill Book Company, 1955. 263 pages, $4.75.
This new edition will appear this spring.

Elementary Differential Equations. By W. T. Martin and Eric Reissner (both of

Massachusetts Institute of Technology). Addison-Wesley Publishing Company, Inc., 1955.

This textbook, to be published this fall, stresses two important separate aspects of the study of differential equations: (1) the formulation of problems in science and engineering as problems of differential equations, and (2) the systematic study of methods of solutions for differential equations arising in applications. An added feature of the book is a chapter on numerical analysis.

Differential Equations. By F. H. Steen (Allegheny College). Ginn and Company, 1955.

This new text, to be published in 1955, will cover all the traditional material. It will be especially designed for a maximum of self-study.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

UNIVERSITY OF OKLAHOMA MATHEMATICS LETTER

The *University of Oklahoma Mathematics Letter* is a four-page publication of interest to high school and beginning college students and teachers. Copies will be sent without charge to persons requesting them and enclosing a stamped, self addressed envelope plus a $3'' \times 5''$ card giving their name and school address. Send all requests to Professor R. V. Andree, Department of Mathematics, The University of Oklahoma, Norman, Oklahoma. You owe it to your students to send for this interesting free publication.

SUMMER INSTITUTES FOR MATHEMATICS TEACHERS

A Summer Institute for Mathematics Teachers will be held in air-conditioned classrooms at the University of Oklahoma, Norman, Oklahoma, from June 6 to June 17, 1955. Teachers may attend either week (one hour credit) or both weeks (two hours credit). The fee is $15 for one week or $25 for two weeks—either credit or non-credit. Official certificates of attendance will be issued. Leaders in the teaching of elementary, junior high school, high school, and beginning college mathematics will be on hand to assist teachers with individual problems related to both classroom procedures and subject matter. Material for the interest and

enrichment of standard courses will be provided. For further information write to F. L. Hayden, Short Courses and Conferences, the University of Oklahoma, Norman, Oklahoma.

The seventh Institute for Teachers and Professors of Mathematics, sponsored by the Association of Teachers of Mathematics in New England, will be held at Middlebury College, Middlebury, Vermont, August 18–25, 1955. Professor J. G. Bowker is General Chairman. For further information write: Miss Harriett Howard, Publicity Chairman, The Ethel Walker School, Simsbury, Connecticut.

SYMPOSIUM ON MATHEMATICAL STATISTICS AND PROBABILITY

The first part of the Third Berkeley Symposium on Mathematical Statistics and Probability was held at the Statistical Laboratory of the University of California, Berkeley, during the last week of December, 1954. Planned as a part of the 121st Annual Meeting of the AAAS, it emphasized applications of statistics and probability. Thus, there were sessions given especially to problems of astronomy, biology, statistical mechanics, industrial research, psychology, and problems of health. One special session was assigned to philosophy of science. Also, there were two sessions given to the theory of statistics.

The second part of the Third Symposium will be held during the forthcoming summer. The second part of the Symposium is planned as a prolonged seminar, extending over July and August, 1955. The central subject of studies will be stochastic processes. The guest speakers include: T. W. Anderson, Columbia University; M. S. Bartlett, University of Manchester; J. Berkson, Mayo Clinic; A. J. L. Blanc-Lapierre, University of Algiers; D. Blackwell, Howard University; J. L. Doob, University of Illinois; W. Feller, Princeton University; R. Fortet, University of Paris; A. Girshick, Stanford University; J. M. Hammersley, Oxford University; W. Hoeffding, University of North Carolina; P. Levy, Paris École Polytechnique; H. Robbins, Columbia University; C. M. Stein, Stanford University.

Because of this event, the summer session program at the Statistical Laboratory will be limited solely to the usual undergraduate courses. The meetings of the Symposium will be open to graduate students and to visiting scholars. Details of the Symposium program may be had by writing to the Statistical Laboratory, University of California, Berkeley 4, California. It is hoped that the *Proceedings of the Third Berkeley Symposium* will be published early in 1956.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1955.

Boston University. May 31 to July 9: Professor Johanson, vector analysis; Professor Scheid, methods of applied mathematics. July 11 to August 20: Dr. Ross, differential equations; Professor Brown, infinite series.

The Catholic University of America. June 27 to August 6: Professor Moller,

higher algebra; Professor Wiegmann, introduction to matrix theory; Professor Taam, advanced calculus I; Professor Rice, advanced calculus II; Professor Finan, basic concepts; Professor Ramler, analytic projective geometry, college geometry, differential equations.

Columbia University, Teachers College. July 5 to August 12: Professor Fehr, teaching arithmetic in the elementary school, applications of geometry; Professors Fehr and Rosskopf, research and departmental seminar in teaching mathematics; Professor Rosskopf, teaching of geometry, foundations of geometry for teachers; Mr. Rourke, business arithmetic in mathematics, professionalized subject matter in advanced secondary school mathematics; Professor Kays, materials and models in mathematics education; Mr. Sobel, supervision and teaching of mathematics in the junior high school.

DePaul University. June 24 to August 4: Professor DeCicco, functions of real variables, principles of geometry; Professor Caton, solid analytic geometry. June 10 to August 3 (evenings): Professor Caton, functions of a complex variable.

George Washington University. June 20 to August 15: Professor Taylor, introduction to boundary value problems, tensor analysis.

Indiana University. June 15 to August 12: Professor Wolfe, theory and application of statistics, non-euclidean geometry; Visiting Professor Polley (Wabash College), history of mathematics, topics for mathematics teachers; Professor MacKenzie, differential equations, algebra and elementary number theory.

Massachusetts Institute of Technology. June 20 to July 1: Professor Hildebrand, numerical analysis.

Michigan State College. June 21 to July 29: Professor Arnold, analysis of variance; Professor J. B. Kelly, theory of equations; Mr. Laurent, correlation analysis; Professor Nordhaus, complex variable I; Professor Olkin, mathematical analysis for the social sciences, factor analysis; Professor Powell, complex variable III; Professor Stewart, college geometry, theory of matrices and groups. June 21 to August 19: Professor Herzog, advanced calculus I, real variable; Professor L. M. Kelly, foundations of mathematics, introductory topology; Mr. Laurent, statistical methods in engineering; Professor Stelson, differential equations, mathematics of engineering; Professor Wells, vector analysis, numerical methods in partial differential equations.

Northwestern University. June 18 to August 13: differential equations, non-euclidean geometry, vector analysis and differential geometry, probability, introduction to the theory of numbers, the continuum, introduction to the theory of groups, engineering mathematics III.

Syracuse University. July 5 to August 13: Professor Baum, introduction to probability; Professor Hemmingsen, intermediate course in algebra; Professor Gelbart, fundamentals of analysis; Professor Exner, heuristic methods in elementary geometry, analysis of elementary mathematics. August 15 to September 16: Dr. Leger, higher mathematics for engineers and scientists I.

University of Buffalo. July 5 to August 12: Professor Gehman, foundations of mathematics; Professor Montague, history of mathematics; Professor Schneckenburger, theory of sets.

University of California. June 20 to July 30: Visiting Professor Steenrod, introduction to algebraic topology.

University of Chicago. June 27 to September 2: A special program will be offered in abstract analysis, including seminars and the following courses: Professor Lashof, introduction to topological algebra; Professor Kaplansky, rings of operators; Visiting Professor Mackey, representation of locally compact groups; Professor Segal, mathematics foundations of quantum mechanics; Professor Halmos, ergodic theory. Other advanced courses are: Professor Chern, differential geometry; Professor Halmos, introduction to modern algebra; Professor Lashof, linear algebra; Professor MacLane, multilinear algebra; Professor Chern, topology and differential geometry of one and two dimensions; Professor Graves, theory of functions of a complex variable.

University of Colorado. June 17 to July 22 and July 25 to August 26: Visiting Professor Hirsch, fundamental ideas in mathematics; Professor Thron, modern algebra; Professor Britton, vector analysis, operational calculus; Professor Hutchinson, functions of a complex variable, partial differential equations.

University of Florida. June 17 to August 13. Graduate Courses in Mathematics: Professor Moore, introduction to mathematical thought; Professor Hadlock, advanced topics in calculus; Professor Rohde, vector analysis; Professor Kokomoor, synthetic projective geometry; Professor Hutcherson, foundations of geometry, differential geometry; Professor Pirenian, Fourier series; Professor Smith, functions of a complex variable; Professor Gager, history of elementary mathematics; Professor South, calculus of variations.

Graduate Courses in Statistics: Professor Marshall, research methods; Professor Harshbarger, design of experiments; Professor Nicholson, theory of probability and theory of sampling; Professor Duncan, advanced statistics; Professors Burrows, Smith and Anderson, special topics; Professor South, advanced topics in calculus; Professor Rulon, educational statistics; Professor Meyer, recent advances in statistics.

The second cooperative Summer Session in Statistics, sponsored by the University of Florida, North Carolina State College, Virginia Polytechnic Institute and the Southern Regional Education Board will be held at the University of Florida from June 20 to July 29. (See this MONTHLY, vol. 61, 1954, p. 65). Inquiries should be addressed to Professor H. A. Meyer, Statistical Laboratory, University of Florida, Gainesville, Florida.

University of Illinois. June 20 to August 13: Professor Day, theory of fields; Professor Ketchum, functions of real variables; Professor Landin, elementary geometry from a modern viewpoint.

University of Michigan. June 20 to August 12: Professor Bott, calculus of variations in the large, functions of a complex variable with applications; Professor Copeland, probability, theory of games; Professor Craig, mathematical

statistics, estimation and significance tests; Professor Hay, advanced mechanics, operational mathematics; Professor Jones, history of algebra, applications of mathematics for teachers; Professor Kaplan, nonlinear differential equations; Professor Leisenring, non-euclidean geometry; Professor LeVeque, advanced calculus; Professor Lyndon, higher algebra; Professor McLaughlin, matrix theory; Professor Nesbitt, mathematics of life insurance, finite differences; Professor Rainich, relativity, vector analysis; Professor Rainville, intermediate differential equations; Professor Rothe, topics in mathematical physics, Fourier series; Professor Samelson, functions of real variables, differential geometry; Professor Young, foundations of mathematics, unified topology.

University of Minnesota, Institute of Technology. June 13 to July 16: Professor Munro, intermediate calculus; Professor Turrittin, variational problems in engineering; Professor Wilcox, vector analysis. July 18 to August 20: Professor Thompson, advanced calculus.

University of Nebraska. June 16 to August 5: Professor Basoco, differential equations; Professor Camp, theory of equations; Professor Jackson, Laplace transforms.

University of North Carolina. June 9 to July 16: Professor Hoyle, elementary algebra from an advanced viewpoint; Professor Winsor, introduction to higher geometry; Professor Garner, history of mathematics; Professor Lasley, analytic geometry from a higher standpoint; Professor Linker, differential equations; Professor Brauer, some recent results in algebra. July 18 to August 24: Professor Hill, elementary mathematical statistics; Professor Cameron, fundamental concepts; Professor Mackie, theory of equations; Professor MacNerny, topics in analysis; Professor Jones, foundations of geometry.

University of Oklahoma. June 10 to August 5: Professor Pan, differential equations; Professor Springer, solid analytical geometry; Professor Bernhart, vector analysis; Professor LaFon, ordinary and partial differential equations; Professor Brixey, theory of groups; Professor Goffman, topics in infinite series.

University of South Carolina. June 10 to August 13: Professor Hedberg, theory of equations, theory of numbers; Professor Novak, introduction to mathematical statistics, college geometry; Professor Williams, advanced calculus, vector analysis.

University of Washington. June 20 to August 19: Professor Kokoris, linear algebra; Professor Jerbert, differential equations; Professor Ballantine, advanced calculus and vector analysis; Professors McFarlan and Cramlet, topics in applied analysis; Professors Dekker and Kingston, advanced euclidean geometry; Professor Beaumont, foundations of mathematics.

University of Wyoming. June 13 to July 15: Professor Varineau, theory of equations, fundamental concepts of mathematics, higher algebra; Professor Schwid, ordinary differential equations; Professor Neubauer, history of mathematics; Professor W. N. Smith, mathematical theory of probability. July 18 to August 19: Professor Barr, advanced calculus, seminar in geometry; Pro-

fessor S. R. Smith, partial differential equations; Professor Steen, college geometry.

West Virginia University. June 6 to July 15: Professor Peters, introduction to algebraic theories, special topics (for teachers); Professor Stewart, higher plane curves. July 18 to August 26: Professor Cunningham, theory of numbers; Professor Posey, group theory, special topics; Professor Vest, advanced differential equations.

PERSONAL ITEMS

University of New Brunswick announces: Dr. J. E. L. Peck, formerly a senior lecturer at the University of Natal, South Africa, has been appointed to an assistant professorship; Associate Professor L. P. Edwards has been promoted to a professorship.

Mr. P. M. Anselone, formerly a mathematician with the General Electric Company, Hanford Works, Richland, Washington, has been appointed Research Associate at the Radiation Laboratory, Johns Hopkins University, Baltimore, Maryland.

Miss Mary N. Applegate, previously a student at the University of Oklahoma, has a position as an engineer at General Electric Company, Schenectady, New York.

Mr. J. D. Armstrong, formerly a junior development engineer for Goodyear Aircraft Corporation, Akron, Ohio, has been appointed to a position as a physics instructor at Bolles School, Jacksonville, Florida.

Dr. E. L. Crow, recently head of the Statistics Branch, United States Naval Ordnance Test Station, China Lake, California, has a position as a statistician in the Office of the Director, Boulder Laboratories, National Bureau of Standards, Boulder, Colorado.

Mr. D. J. Davis, previously a physicist with the Rocket Development Group, Redstone Arsenal, Huntsville, Alabama, has accepted a position as project engineer with the R. C. A. Service Company, Missile Test Project, Patrick Air Force Base, Florida.

Assistant Professor Mamie M. Davis of Francis T. Nicholls Junior College of Louisiana State University has been appointed to an associate professorship at Wesleyan College.

Mr. August Deckert, formerly a mathematician at White Sands Proving Grounds, Las Cruces, New Mexico, has a position as a research engineer with Boeing Airplane Company, Seattle, Washington.

Mr. W. C. Dixon, previously a research associate at the Computation Laboratory, Wayne University, is now a member of the Computer Systems Division, Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. J. R. Fleming, recently a graduate student at the University of Washington, has a position at the Radiation Laboratory, University of California, Livermore.

Mr. J. B. Freier of St. Peter's College has been appointed to an assistant professorship at the Rensselaer Polytechnic Institute.

Mr. B. A. Fusaro has been appointed to an instructorship at Middlebury College.

Dr. L. V. Good, formerly a graduate student at the State College of Washington, is now Director of Anchorage Community College, Alaska.

Professor Emeritus G. H. Graves of Purdue University is a visiting lecturer at Valparaiso University.

Mr. J. B. Hansen, previously a student at Harvard University, has a position as a mathematician with the Computer Control Company, Naval Air Missile Test Center, Point Mugu, California.

Assistant Professor Katharine E. Hazard of New Jersey College for Women, Rutgers University, has been promoted to an associate professorship.

Mr. Peter Henrici of American University has been promoted to an assistant professorship.

Mr. J. G. Horne, Jr., has been appointed to an instructorship at Tulane University.

Reverend Brother Calixtus James, recently at LaSalle Military Academy, Oakdale, New York, is now at Bishop Bradley High School, Manchester, New Hampshire.

Assistant Professor J. A. Jenkins of Johns Hopkins University has been appointed to an associate professorship at the University of Notre Dame.

Mr. J. S. Klein, formerly a research assistant at the University of Michigan, has been appointed to an instructorship at Oberlin College.

Mr. A. H. Kruse, previously a graduate student at the University of Chicago, has been appointed to a research instructorship at the University of Kansas.

Mr. L. H. Lange of Valparaiso University has been promoted to an assistant professorship.

Mr. R. E. Lewkowicz, previously in military service, has a position as an associate mathematician with the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Dr. Mark Lotkin, formerly chief of Machines Section, Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland, is now Manager, Computer Mathematics Unit, R. C. A. Service Company, Patrick Air Force Base, Florida.

Mr. L. C. Marshall, previously a mathematician for White Sands Proving Ground, Las Cruces, New Mexico, is a computing analyst for North American Aviation, Downey, California.

Mr. R. M. Meisel of New York University has accepted a position as Applied Mathematical Aerodynamicist with Republic Aviation Corporation, Farmingdale, New York.

Mr. R. A. Moreland, Jr., formerly a teaching fellow at Texas Technological

College, has a position as a mathematician with the Sandia Corporation, Albuquerque, New Mexico.

Professor Zeev Nehari of Washington University has been appointed to a professorship at Carnegie Institute of Technology.

Associate Professor C. L. Perry of the United States Naval Postgraduate School, Monterey, California, has been promoted to a professorship.

Dr. J. H. Powell, previously a graduate research assistant at Michigan State College, has been appointed to an instructorship at the University of Detroit.

Mr. John J. Quinn, formerly a student at St. John's University, New York, is now a graduate assistant in the Department of Physics, University of Maryland.

Assistant Professor Anatol Rapoport of the University of Chicago is a fellow at the Center for Advanced Study in the Behavioral Sciences, Stanford, California.

Mr. E. A. Rasor, recently an actuarial mathematician for the Social Security Administration, Washington, D. C., has accepted a position as an actuary for the Retirement Division of the United States Civil Service Commission, Washington, D. C.

Dr. B. A. Rattray, formerly a lecturer at the University of New Brunswick, has been appointed to an assistant professorship at McGill University.

Dr. E. K. Ritter, Director of Computation and Ballistics, United States Naval Proving Ground, Dahlgren, Virginia, has accepted an appointment as Professor and Director of the Rich Electronic Computer Center, Engineering Experiment Station, Georgia Institute of Technology.

Miss Mildred G. Schlapkohl, previously a student at Northwestern University, is now an assistant mathematician at Argonne National Laboratory, Lemont, Illinois.

Dr. C. V. L. Smith, recently head of Computer Branch, Office of Naval Research, Washington, D. C., is now a scientific liaison officer at the Office of Naval Research (London Branch), New York City.

Mr. H. F. Smith, formerly an instructor at Elgin High School and Community College, Illinois, is now an applied science representative with the I. B. M. Corporation, Chicago, Illinois.

Mr. E. A. Stavinoha, previously a mathematics teacher at Brownsville High School, Texas, has accepted a position as a computing analyst for Douglas Aircraft Company, Tulsa, Oklahoma.

Mr. Isay Stemp is Assistant Director of Research for J. A. Deknatel & Son, Inc., Queens Village, Long Island, New York.

Mr. A. D. Stewart of the University of Wisconsin has been appointed to an assistant professorship at the Prairie View Agricultural and Mechanical College, Texas.

Miss Roberta A. Wilcox, formerly a student at the University of Rhode Is-

land, is now a research trainee in the Department of Biostatistics, School of Hygiene and Public Health, Johns Hopkins University.

Mr. R. F. Willis, previously a mathematician with the Operations Research Group of Arthur D. Little, Inc., Cambridge, Massachusetts, has a position as a mathematician with the Weapons Systems Laboratory, Aberdeen Proving Ground, Maryland.

Mr. W. H. Winnis, formerly a junior actuary with the Union Labor Life Insurance Company, New York City, has accepted a position as an actuarial assistant for the Insurance Department, State of Maryland, Baltimore, Maryland.

Dr. R. A. Zemlin, previously a National Science Foundation fellow at Ohio State University, is now a senior mathematician at Remington Rand Corporation, St. Paul, Minnesota.

Professor Irvin S. Cohen of the Massachusetts Institute of Technology died on February 14, 1955.

Professor Emeritus Edward Kasner of Columbia University died on January 7, 1955. He was a charter member of the Association.

Professor D. W. Pugsley, head of the Department of Mathematics of Berea College, died on December 15, 1954. He was a member of the Association for 27 years.

Professor Emeritus C. N. Reynolds of West Virginia University died on December 15, 1954. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE TREASURER FOR THE YEAR 1954

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1954. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

At the end of the year, the total funds of the Association included balances held for the Committee on Visiting Lecturers and the Committee on the Undergraduate Program, most of which will be expended during 1955.

During 1954, the Association paid its share of the expenses of the Combined Membership Lists for 1952 and 1953. Since a large part of the cost of the 1952 List was a capital expense, the Board of Governors voted to appropriate $2,000 from the Dunkel Fund for this item.

The Current Fund suffered an actual deficit of $5,134 during 1954, as compared with a surplus of $393 in 1953. Most of this deficit is due to the charges for the Combined Membership List. It is the opinion of the Finance Committee that either the publication of the List should be discontinued or that it should be financed entirely by one of the special funds of the Association.

I. Total Funds of the Association on January 1, 1954

M & T Trust Co., Buffalo	$11,124.99	Current Fund	$11,124.99
Securities	83,283.00	Carus Fund	14,833.38
		Chace Fund	12,979.19
		Houck Fund	8,792.72
		Chauvenet Fund	1,115.58
		Dunkel Fund	16,373.09
		General Fund	28,189.04
	$94,407.99		$94,407.99

II. Current Fund

Balance, January 1, 1954	$11,124.99	Monthly	
Dues	21,199.59	Publication	$17,699.29
Initiation fees	696.00	Reprints	572.53
Subscriptions	5,110.34	Editor's Office	1,758.39
Sale of back numbers	1,665.73	Secretary-Treasurer's Office	
Advertisements	3,577.50	Clerical help	7,982.84
Contributions	3.00	Postage and printing	1,721.27
Sale of exchange periodicals	150.80	Office expenses	518.65
Interest on General Fund	1,171.49	Bank fees	215.82
Income from Hardy Fund	120.00	Auditing fee	50.00
Charges against Funds	233.46	Board of Governors	1,766.96
		Meetings	822.16
		Committees	1,000.00
		Subventions	505.17
		1952 Combined Membership List.	1,290.51
		1953 Combined Membership List.	3,158.88
		Transfer to General Fund	5,494.81
		Balance, December 31, 1954	$ 495.62

	III. Carus Fund	IV. Chace Fund	V. Houck Fund
Balance, January 1, 1954	$14,833.38	$12,979.19	$ 8,792.72
Sale of publications	1,582.19	731.63	20.80
Interest	595.79	522.15	354.80
Increase in value of securities	2,881.08	2,525.00	1,715.70
National Research Council grant	—	1,000.00	—
Less: charges and bank fees	245.40	178.87	53.99
Fire Insurance	21.00	—	—
Printing	—	2,781.63	—
Balance, December 31, 1954	$19,626.04	$14,797.47	$10,830.03

	VI. Chauvenet Fund	VII. Dunkel Fund	VIII. General Fund
Balance, January 1, 1954	$ 1,115.58	$16,373.09	$29,189.04
Interest	43.51	659.38	—
Increase in value of securities	210.42	3,188.62	5,665.06
Dunkel Estate payment	—	240.80	—
Transfer from Current Fund	—	—	5,494.81
Less: charges and bank fees	6.37	96.48	—
Mathematics Student Journal	—	500.00	—
1952 Combined Membership List	—	2,000.00	—
Balance, December 31, 1954	$ 1,363.14	$17,865.41	$40,348.91

	IX. Visiting Lecturers Fund	X. Fund for Committee on Undergraduate Program
National Science Foundation grant	$15,000.00	—
Social Science Research Council grant	—	$2,500.00
Transfer from Current Fund	500.00	500.00
Contributions toward travel expenses	505.00	—
Less: salaries and travel expenses	5,393.36	2,432.26
Committee expenses	149.36	58.58
Balance, December 31, 1954	$10,462.28	$ 509.16

XI. Total Funds of the Association on December 31, 1954

Current Fund	$ 495.62	M & T Trust Co., Buffalo	$ 11,467.06
Carus Fund	19,626.04	Securities	104,831.00
Chace Fund	14,797.47		
Houck Fund	10,830.03		
Chauvenet Fund	1,363.14		
Dunkel Fund	17,865.41		
General Fund	40,348.91		
Visiting Lecturers Fund	10,462.28		
Fund for Committee on Und. Program	509.16		
	$116,298.06		$116,298.06

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 97 persons have been elected to membership by the Board of Governors on applications duly certified.

D. S. ADORNO, B.A. (Texas) Senior Mathematician, Haller, Raymond & Brown, State College, Pa.

N. W. ALBRIGHT, Student, California Institute of Technology.

D. R. ARNOLD, Student, Bethany College.

F. A. BARAGAR, Student, University of Manitoba.

C. W. BARNES, A.B. (High Point) Part-time Instr. and Grad. Student, University of North Carolina.

C. B. BAYTOP, M.S. (Howard) Instr., Hampton Institute.

A. P. BERENS, Student, University of Dayton.

LIPMAN BERS, Dr.Rer.Nat. (Prague) Professor, New York University.

S. E. BOHN, M. A. (Nebraska) Asst. Professor Concordia College.

J. R. BROWN, B.A. (Oregon S.C.) Grad. Student, Oregon State College.

C. E. BURGESS, Ph.D. (Texas) Asst. Professor, University of Utah.

R. R. CHRISTIAN, Ph.D. (Yale) Instr., University of British Columbia.

DR. J. B. COMROE, M.D. (Chicago) Physician, Berkeley, Calif.

B. R. DAVIS, B.S. (Geneva) Grad. Asst., University of Pittsburgh.

J. F. DAVIS, B.A. (Pennsylvania S.U.) Quality Control Engr., Firestone Tire and Rubber Co., Akron, Ohio.

E. R. DEAL, M.S. (Kansas S.C.) Instr., University of Michigan.

E. C. DELAND, M.A. (U.C.L.A.) Grad. Student, University of California at Los Angeles.

ANDREW DELANEY, B.A. (Oberlin) Vice President and Actuary, American General Life Insurance Co., Houston, Texas.

MRS. MARTHA M. DE VALLE, M.S. (Mexico) Research Worker, Institute of Geophysics, University of Mexico.

J. O. DISTAD, B.S. (Montana S.C.) Grad. Student, Montana State College.

T. H. DYER, B.S. (U.S. Naval Academy) Captain, United States Navy, Washington, D. C.

COL. E. H. EDDY, M.S. in A.E. (Calif. I.T.) Instr., General Motors Institute, Flint, Mich.

K. H. ERDODY, M.S. (Michigan) Instr., Wayne University; Northern High School, Detroit, Mich.

CATHERINE M. FITZPATRICK, Mathematician, Rock Island Arsenal, Rock Island, Ill.

J. E. FLANAGAN, Ph.D. (Illinois) Applied Science Representative, I.B.M., Pittsburgh, Pa.

PAUL FONG, Student, University of California, Berkeley.

OLNEY FORTIER, Danielson, Conn.

Y. T. FUNG, Student, University of California, Berkeley.

B. A. FUSARO, M.A. (Columbia) Instr., Middlebury College.

M. L. GLASSER, B.A. (Chicago) Grad. Student, University of Chicago.

WILLIAM GOOGE, Student, Georgetown University.

R. P. GOSSELIN, Ph.D. (Chicago) Professor, Youngstown College.

R. G. GREEN, Student, Abilene Christian College.

J. G. HARVEY II, Student, Baylor University.

COL. ARCHIE HIGDON, Ph.D. (Iowa S.C.) Head, Department of Mathematics, United States Air Force Academy, Lowry Air Force Base, Denver, Colo.

G. W. HILLER, Student, Mississippi Southern College.

M. L. HODGINS, Student, Sacramento State College.

C. A. HOFFMAN, M.S. (Rutgers) Instr., University of Florida.

J. T. HUMPHREY, M.S. (S.U. of Iowa) Asso. Professor, Grambling College.

R. Q. JENNETT, M.S. (Purdue) Aerophysics Engr., Consolidated Vultee Aircraft Corp., Ft. Worth, Texas.

EDGAR KARST, M.S. (Breslau) Independence, Missouri.

A. G. KONHEIM, Student, Polytechnic Institute of Brooklyn.

W. A. KRZEMINSKI, B.S. in Ed. (Bucknell) Instr., Wallington High School, N. J.

C. M. LARSEN, A.M. (Cornell) Instr., San Jose State College.

T. L. LIBBY, Student, Baylor University.

J. H. LINDSAY, JR., Student, University of Toronto.

R. D. LOW, M.S. (Iowa S.C.) Private, United States Army.

E. S. LOWRY, Student, University of Toronto.

MRS. RUTH J. MACKICHAN, M.A. (Michigan) Instr., University of North Dakota.

MARTHA J. MAY, Student, University of Kentucky.

R. M. MEISEL, M.S. (N.Y.U.) Aerodynamicist, Republic Aviation Corp., Farmingdale, N. Y.

KATHRINE C. MIRES, M.A. (Arkansas) Asst. Professor, Northwestern State College, Alva, Okla.

J. E. MULLINS, Student, Siena College.

HERBERT NADLER, M.A. (Columbia) Corporal, United States Army.

J. L. NEFF, B.S. in Ed. (Otterbein) Instr., University of Dayton.

W. C. OLSON, M.S. (Delaware) Mathematician, Aberdeen Proving Ground

E. T. PARKER, B.A. (Northwestern) Assistant, Ohio State University.

C. M. PEARCY, JR., B.A. (Texas A. & M.) Teaching Fellow, A. & M. College of Texas.

JACQUELINE L. PENEZ, Ph.D. (Minnesota) Instr., Barnard College, Columbia University.

J. B. PHARR, Student, Davidson College.

LT. A. H. PUCCI, B.S. (New Hampshire) Mathematician, Wright-Patterson Air Force Base, Ohio.

GUSTAVE RABSON, Ph.D. (Michigan) Asst. Professor, Antioch College.

E. A. RACE, M.A. (Maine) Asso. Professor, Norwich University.

DAVID ROSEN, Ph.D. (Pennsylvania) Asst. Professor, Swarthmore College.

MORRIS ROSEN, Student, Hofstra College.

W. C. ROSS, JR., Ph.D. (S.U. of Iowa) Instr., Knox College.

ANNE RYCHLICKI, Student, Alliance College.

P. P. SAWOROTNOW, Ph.D. (Harvard) Instr., Catholic University of America.

D. H. SCHMIEDER, A.B. (Kentucky) Grad. Asst., University of Kentucky.

M. M. SEGAL, M.S. in A.E. (N.Y.U.) Special Lecturer, Newark College of Engineering.

M. A. SELIM, M.S., M.A. (Texas) Research Asso., Petroleum Research Co., Denver, Colo.

DANIEL SHANKS, Ph.D. (Maryland) Acting Chief, Applied Mathematics Division, Naval Ordnance Laboratory.

A. B. SHEAFFER, M.A. (Sacramento S.C.) Major, United States Air Force.

W. C. SIKES, M.A. (Colorado) Asst. Professor, Abilene Christian College.

SISTER JULIA, M.S. (Catholic) Instr., College of Mount St. Joseph.

SISTER M. CELINE, Ph.D. (Michigan) Professor, Mercyhurst College.

SISTER MARY LORETTA, Student, Mercy College.

SISTER REGINA MARY, A.B. (Mount St. Joseph) Instr., College of Mount St. Joseph.

SISTER RUTHMARY, B.S. (La Crosse S.C.) Head, Mathematics Department, Viterbo College.

J. R. SLAGLE, Student, St. John's College, Brooklyn, N. Y.

H. R. SMITH, M.S. (South Carolina) Head, Mathematics Department, Baltimore Polytechnic Institute.

W. G. SPOHN, JR., M.A. (California) Instr., University of Delaware.

A. D. STEWART, M.S. (Howard) Asst. Professor, Prairie View A. & M. College.

W. C. SWIFT, B.S. (Kentucky) Grad. Student, University of Kentucky.

L. R. TAPPAN, M.A. (Arkansas) Instr., University of Arkansas.

W. P. UDINSKI, Ph.D. (Illinois) Mathematician, U.S. Air Force, Lackland Air Force Base, Texas.

MILDRED J. WESEEN, M.A. (California) Head, Mathematics Department, Analy Union High School, Sebastopol, Calif.

NANCY J. WHITE, B.S. (Millersville S.T.C.) Teacher, Edward Hand Junior High School, Lancaster, Pa.

R. J. WISNER, Ph.D. (Washington) Asst. Professor, Haverford College.

H. A. WOLF, B.S. (Illinois) Grad. Student, DePaul University.

F. W. WOLOCK, M.S. (Catholic) Instr., Iona College.

GREGORY WULCZYN, M.A. (Pennsylvania) Instr., Bucknell University.

KOICHI YAMAMOTO, M.S. (Tokyo) Asst. Professor, Kyushu University, Fukuoka, Japan

C. T. YANG, Ph.D. (Tulane) Member, Institute for Advanced Study.

B. H. YOUELL, JR., B.S. (W. Va. I. T.) Grad. Asst., West Virginia University.

L. N. ZACCARO, M.S. (Connecticut) Asst. Instr., Syracuse University.

ROMUALD ZALUBAS, M.A. (Vytautas) Instr., Georgetown University.

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Princeton University, Princeton, New Jersey, on November 27, 1954. Professor Alexander Tartler, Chairman of the Section, presided. There were present one hundred fifteen individuals including seventy-eight members of the Association:

J. H. Barrett, Donald C. Benson, J. S. Biggerstaff, B. H. Bissinger, Leila Dragonette Bram, H. W. Brinkmann, R. C. Carson, E. H. Cutler, L. J. Deck, F. L. Dennis, G. F. Feeman, N. J. Fine, R. F. Gabriel, E. D. Glenney, G. B. Glover, S. I. Goldberg, H. H. Goldstine, D. S. Greenstein, V. H. Haag, Theodore Hailperin, Katherine E. Hazard, S. A. Hoffman, J. R. Holzinger, C. C. Hsuing, R. F. Jackson, R. C. James, C. E. Kerr, R. W. Klopfenstein, T. L. Koehler, R. J. Kohlmeyer, J. B. Kruskal, Jr., M. D. Kruskal, Harold W. Kuhn, R. A. C. Lane, V. V. Latshaw, W. S. Lawton, Solomon Lefschetz, Marguerite Lehr, W. H. Leser, A. W. Mall, Helen M. Marston, B. H. McCandless, William McKay, S. S. McNeary, Josephine M. Mitchell, E. R. Mullins, Jr., C. A. Nelson, C. O. Oakley, Marghrita L. O'Neil, J. C. Oxtoby, R. S. Pieters, C. F. Pinzka, G. E. Raynor, Edgar Reich, Dorothy J. Rhea, B. E. Rhoades, David Rosen, F. E. Ross, C. W. Saalfrank, I. J. Schoenberg, Pincus Schub, Jerome C. Smith, Ernst Snapper, Sister Mary Stephanie, E. P. Starke, J. D. Swift, Alexander Tartler, C. M. Terry, D. L. Thomsen, Jr., Bryant Tuckerman, R. M. Walter, G. C. Webber, D. W. Western, Anna Pell Wheeler, M. E. White, Albert Wilansky, F. H. M. Williams, Marie A. Wurster.

Officers of the Section for the year 1954–55 are: Chairman, Professor T. L. Koehler, Muhlenberg College; Secretary, Professor G. C. Webber, University of Delaware; Program Committee: Chairman, Professor C. W. Saalfrank, Lafayette College; Professor N. J. Fine, University of Pennsylvania; Professor H. S. Grant, Rutgers University.

The following papers were presented:

1. *Coordinates of algebraic varieties*, by Professor Ernst Snapper, University of Southern California and Princeton University.

Let $\{V_\sigma^m\} = A$ be a set of pure, algebraic varieties of dimension m and degree g, imbedded in an n-dimensional projective space S^n over an arbitrary groundfield k. The irreducible components of each V_σ^m all have dimension m, and are counted with preassigned multiplicities. Such pure varieties, whose components are furnished with multiplicities, are called cycles of dimension m and degree g. We say that A is an *algebraic system* of cycles if, in some projective or multiply projective space, there exists an algebraic variety P and an algebraic correspondence T from P to S^n, satisfying the following conditions: (1) T associates to each point of P exactly one cycle of A, (2) T associates to

different points of P, different cycles of A, (3) every cycle of A is the image under T of a point of P.

If both (P, T) and (P', T') satisfy the above conditions, there exists a birational transformation R from P onto P' where both R and R^{-1} are regular and where $T'R = T$; namely, if $p \in P$, $R(p)$ is defined by $T'(R(p)) = T(p)$. Hence, for all purposes of algebraic geometry, (P, T) and (P', T') are equivalent. These conventions free us from the unnecessary and very undesirable restriction of always having to construct Chow-varieties in a preassigned way. When (P, T) satisfies the above conditions P is called a Chow-variety of A, a point $p \in P$ is called a Chow-point of the cycle of A which it represents, and the coordinates of P are called the coordinates of that cycle. The dimension of P is termed the dimension of A and, in the same way, all of the algebra-geometric terminology is carried over from P to A.

Several examples were given. In each example, P and T were constructed precisely. As an application it was proved that, on a locally normal variety, a linear system is always contained in a complete linear system.

2. *Mathematics through the television lens*, by Professor F. G. Fender of Rutgers University, Professor Marguerite Lehr of Bryn Mawr College, and Professor R. F. Jackson, University of Delaware.

Professor Fender reported on a thirteen-week series of talks entitled "This is Mathematics" which was presented over television by Rutgers University. The series was aimed at making clearer to the general public the story of the modern mathematician and his work. Topics of the individual talks were Counting, Number Systems, Adding and Multiplying, Subtracting and Dividing, Number Theory, Machines, Greatest and Least, The Shape of Things, Uncertainty, Symbols, The Possible and Impossible, Audience-suggested Program.

Professor Lehr described a fifteen-week series of half-hour talks which was presented over Philadelphia's Channel 6, accompanied by a semester syllabus of abstracts and references for each topic. See this MONTHLY, vol. 62, 1955, pp. 15–21.

Professor R. F. Jackson presented an outline of the general philosophy and specific plans for a television series on "Thinking Machines—from Fingers to Flip-Flops."

3. *Area and volume*, by Professor A. S. Besicovitch, Trinity College, Cambridge University, England, and the Institute for Advanced Study, introduced by the Secretary.

A topological transform of a square in a plane or of a cylinder in 3-space is called a topological square or topological cylinder. The transforms of sides, vertices and bases are called sides, vertices and bases of the topological figure. The S-distance between two points of a topological figure is defined as the minimum length of curves joining them and entirely belonging to the topological figure. It is shown: (1) The area of a topological square is greater than or equal to ah, when a and h are the distances between pairs of opposite sides; (2) The volume of a topological cylinder whose height is h and for which the area of any side section is greater than a need not be greater than or equal to ah.

4. *Some remarks on numerical stability*, by Dr. H. H. Goldstine, Electronic Computer Project, Institute for Advanced Study.

The concept of numerical stability plays a fundamental role in modern day numerical analysis. Since machines do not multiply or divide with exactitude, it is of importance to estimate how these errors accumulate in the course of a calculation. Examples were given showing how round-off errors could accumulate in such a fashion that no significance could be attached to the final results.

Following the program the Electronic Computer at the Institute was demonstrated.

G. C. WEBBER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 7, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May 7, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS

KENTUCKY, Georgetown College, Georgetown, April 30, 1955.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Morgan State College, Baltimore, Maryland, April 16, 1955.

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN

MINNESOTA, College of St. Teresa, Winona, Minnesota, May 7, 1955.

MISSOURI, University of Kansas City, April 22, 1955.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1955.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 23, 1955.

OKLAHOMA

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.

PHILADELPHIA

ROCKY MOUNTAIN, University of Wyoming, Laramie, April 22–23, 1955.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN, University of New Mexico, Albuquerque, April 8-9, 1955.

TEXAS, Abilene Christian College, Abilene, April 22–23, 1955.

UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.

WISCONSIN, Cardinal Stritch College, Milwaukee, May 7, 1955.

THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9 and 10, additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. The more recent numbers are:

No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.

No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.

No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.

No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.

No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.

No. 8. *Rings and Ideals* by N. H. McCoy, xii +216 pages.

No. 9. *The Theory of Algebraic Numbers* by Harry Pollard, xii+143 pages.

No. 10. *The Arithmetic Theory of Quadratic Forms* by B. W. Jones, x+212 pages.

COLLEGE ALGEBRA

by MOSES RICHARDSON, Brooklyn College

This extremely lucid text gives unusual insight into sound mathematics. It is adaptable to classes of any degree of preparation and combines careful explanation of procedure with reasonable motivation for the student. Your students will not find it excessively rigorous—where correct proof would be difficult, a searching discussion is substituted. The difficult sections have been starred both in the text itself and in the Table of Contents. (Those exercises which are related to starred topics have also been starred.)

Illustrative material is used to bring out the practical aspects of college algebra.

472 pages *6" x 9"* *Published 1947*

PLANE TRIGONOMETRY, 3rd Edition

by FRED W. SPARKS, Texas Technological College and PAUL K. REES, Louisiana State University

Here is a book—written for college freshmen—which covers all the essentials of Plane Trigonometry, including logarithms, graphs of the trigonometric functions and trigonometric equations. This 3rd Edition contains more problems than the previous one (over 1800 as compared with 1400). Also, a new arrangement of problems enables the instructor to make a desirable assignment without having to study through the exercises, and also assures adequate coverage of the material by applying 4 entirely different assignments.

With tables—275 pages *6" x 9"* *Published 1952*
Without tables—199 pages

CALCULUS, 3rd Edition

by GEORGE E. F. SHERWOOD, UCLA and ANGUS E. TAYLOR, UCLA

Here is an improved edition of one of the leading sellers in the calculus. (Previous editions sold over 100,000 copies.) An outstanding feature of this 3rd Edition is the very early introduction of the inverse of differentiation, in Chapter II, items 16, 17 and 18. The actual techniques here are limited to polynomials.

Of particular interest is a section on applications to problems of velocity and acceleration in rectilinear motion. Students who take physics concurrently with the beginning calculus, engineering students, for example, will find this extremely useful.

Answers to exercises and problems free on adoption.

579 pages *6" x 9"* *Published 1954*

For approval copies write

PRENTICE-HALL, Inc. • 70 FIFTH AVENUE, NEW YORK 11, N.Y.

New Ginn Publications

TAYLOR: ADVANCED CALCULUS

A thorough treatment of principles with emphasis on sound understanding of concepts. By Angus E. Taylor, University of California, Los Angeles

STEEN: DIFFERENTIAL EQUATIONS

Designed for a maximum of self-study, this text treats both ordinary and partial differential equations. By Frederick H. Steen, Allegheny College

GINN AND COMPANY 1867

Home Office: Boston *Sales Offices:* New York 11 Chicago 16
Atlanta 3 Dallas 1 Columbus 16 San Francisco 3 Toronto 7

Creative Engineering Opportunities with Republic

Dynamics

Senior Mathematics

Outstanding opportunity for a person with real mathematical ability. Here's your chance to build a firm professional background in Applied Mathematics by working with Engineers in the Mathematical formulation of complex problems for high speed digital computers. Dealing with the complex problems associated with supersonic aircraft is a real professional challenge and the person with outstanding ability can earn rapid, solid advancement. An M.A. or Ph.D. in Mathematics is required and computing experience desirable but not essential.

Flutter & Vibration

Dynamics Engineer

Responsible position open for work in flutter analysis of supersonic aircraft. Good environment of qualified people and adequate equipment provide means for gaining valuable experience in the field of aeronautical engineering. Mathematics or engineering degree required. Very good opportunity for advancement.

Please address complete resume, outlining details of your technical background, to:

Mr. R. L. Bortner
Administrative Engineer

REPUBLIC AVIATION
FARMINGDALE, LONG ISLAND, NEW YORK

Ready in April

A NEW TEXT FOR ALGEBRA AND TRIGONOMETRY COURSES

COLLEGE ALGEBRA AND PLANE TRIGONOMETRY

By ABRAHAM SPITZBART *and* ROSS H. BARDELL, *University of Wisconsin*

This new textbook offers a careful, thorough treatment of college algebra and plane trigonometry, combined in one volume for maximum teaching effectiveness and economy of presentation. Integration of the two subjects has been planned to emphasize the trigonometric functions and other functions which are studied, but artificial integration, merely for the sake of integrating, has been avoided. Unification of algebraic and trigonometric ideas is stressed in illustrative examples and problems.

The outstanding feature of the book is its use of the function concept as the unifying theme, a device which has met with wide favor in the authors' COLLEGE ALGEBRA. The first four chapters offer a review of high school algebra, and the remainder of the book is devoted to the presentation of college algebra and trigonometry. The text is so planned that each exercise group, together with the textual material preceding it, constitutes a logical teaching unit.

COLLEGE ALGEBRA AND PLANE TRIGONOMETRY is designed for a four or five semester-hour course, but is also suitable for a full two-semester course for students with less adequate preparation.

Cloth, c. 284 pp., 70 illus., 1955 — $4.50

BY THE SAME AUTHORS

COLLEGE ALGEBRA

By ROSS H. BARDELL *and* ABRAHAM SPITZBART

Already the adopted text in nearly fifty colleges and universities, this new book may be used as a text for terminal students or for those who will continue in mathematics. The book is organized around the concept of function. After the early introduction of this concept, all succeeding material is related to it wherever possible.

Excerpts from a review in the AMERICAN MATHEMATICAL MONTHLY:

"It is a rare and welcome pleasure, these days, to encounter the slim and compact freshman algebra text that achieves so admirably the goals intended by the authors. It is an extremely flexible and teachable text which allows of departures and amplifications without upsetting the continuity of the material. Clearly presented, concisely written, uncluttered by the irritating confusions of overcautious development, yet it lacks none of the rigor necessary to such a text.

"Probably the most exciting chapter in this fine little book is the one on Mathematical Induction and the Binomial Theorem. The method of induction is very lucidly presented and numerous examples are provided whereby the student may gain confidence in the use of this method of proof."

HAVE YOU SEEN A COPY OF COLLEGE ALGEBRA?

Cloth, 248 pp., 22 illus., 1st ed., 2nd ptg. 1954 — $3.75

EXAMINATION COPIES AVAILABLE UPON REQUEST

ADDISON-WESLEY PUBLISHING COMPANY, INC., Cambridge 42, Massachusetts

Standard texts by Lloyd L. Smail

ANALYTIC GEOMETRY AND CALCULUS

Designed for college courses which combine a study of calculus with that of analytic geometry, this book emphasizes the meaning of fundamental concepts and states important formulas in the form of theorems, giving the conditions of validity and meaning of the symbols involved. A logical organization is followed, beginning with sufficient analytic geometry to prepare the student for the early parts of calculus. Additional topics of analytic geometry are introduced as they are needed for gaining an understanding of the calculus. An abundance and variety of exercises are provided.

Large Octavo ***672 pages*** ***diagrams*** ***$5.50***

CALCULUS

Outstanding in its field, this text for college courses in Calculus offers a variety of distinctive features. Noteworthy among these are the early introduction of integration, involving both indefinite integrals and definite integrals; the replacement of Duhamel's theorem with a simpler theorem due to Bliss; the modern definition of limit of a function without defining limit of a variable; the analytic proof of the fundamental theorem of integration; and the early introduction of hyperbolic functions.

Large Octavo ***592 pages*** ***diagrams*** ***$5.00***

Appleton Century Crofts Inc.

Publishers of
THE NEW CENTURY CYCLOPEDIA OF NAMES

35 West 32nd St. New York

2 Creative Mathematicians

● **NEED . . .**

Two men for development of techniques for solution of critical technical problems of stress and vibration in preliminary design of aircraft jet engines.

● **THEY WILL . . .**

initiate and co-ordinate analytically formulated tests involving laboratory equipment, actual engines and engine components.

● **MUST HAVE . . .**

B.S. in engineering or physics and advanced degree in mathematics. Should have experience in applying mathematics to machine design.

Send Resume . . .

GIVING COMPLETE INFORMATION AS TO EDUCATION & EXPERIENCE

Box 47, American Mathematical Monthly
University of Rochester, Rochester, N.Y.

HEATH College Texts

by WILLIAM L. HART

Calculus

Ready this spring

This new course in calculus offers a thorough, modern presentation, rigorous, yet skillfully adapted to student understanding. Organization, textual exposition, examples, exercises, and problems of widely varied application make this text a teaching instrument of the highest quality. Answers are provided in the book for all odd-numbered problems; answers for even-numbered problems are provided free in a separate pamphlet. Publication about April 15. $5.50.

College Algebra

FOURTH EDITION

Emphasizing modern viewpoints, this text features the tested sequence of topics and pedagogical methods of earlier editions. The book contains a novel introduction to signed numbers; a unique chapter on discrete probability; and appendix notes on sets, with probability contacts. Answers are provided in the book for all odd-numbered problems; answers for even-numbered problems are provided free in a separate pamphlet. 420 pages of text. $3.75.

D. C. Heath and Company

SALES OFFICES: NEW YORK 14 CHICAGO 16 SAN FRANCISCO 5 ATLANTA 3 DALLAS 1

HOME OFFICE: BOSTON 16

Number 33, Annals of Mathematics Studies

CONTRIBUTIONS TO THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

Edited by L. BERS, S. BOCHNER, and F. JOHN

The results of a three day conference on partial differential equations held at Arden House, Harriman, New York in October 1952. Contributors include: S. Bergman, L. Bers, S. Bochner, F. E. Browder, J. B. Diaz, A. Douglis, F. John, P. D. Lax, J. Leray, C. Loewner, A. N. Milgram, C. B. Morrey, L. Nirenberg, M. H. Protter, P. C. Rosenbloom. *Annals of Mathematics Studies, #33.*

272 pages, $4.

Order from your bookstore, or

PRINCETON UNIVERSITY PRESS, Princeton, New Jersey

the RCA COMPUTER ENGINEER

RCA's continuous electronic research and development provide immediate openings in its Digital Computer Activity for EE's, ME's, Physicists and Mathematicians. Junior and Senior openings in all fields.

These positions will offer a challenge to your skill and creative ability. Openings are in the Greater Philadelphia area.

There are many opportunities in:

- *PROGRAMMING*
- *INPUT-OUTPUT DEVICES*
- *VIDEO PULSE TECHNIQUES*
- *MAGNETIC RECORDING*
- *HIGH SPEED MEMORY*
- *FEED BACK AMPLIFIERS*
- *CODING AND DECODING*
- *RELAY CONTROL*

Send a complete resume of your education and experience to:

Mr. John R. Weld, Employment Manager
Dept. A-8D, Radio Corporation of America
Camden 2, New Jersey

RADIO CORPORATION OF AMERICA

WILEY — BOOKS

AN INTRODUCTION TO DEDUCTIVE LOGIC

By Hugues Leblanc, Bryn Mawr College. Covers a large amount of primary material, including eight groups of calculi, and maintains an equal balance between the philosophy and the mathematics of logic. Featured is a dual approach to the logic of statements and the logic of quantifiers. Numerous exercises accompany the text. 1955. 244 pages. $4.75.

ADVANCED MATHEMATICS FOR ENGINEERS, Third Edition

By H. W. Reddick, New York University, and F. H. Miller, The Cooper Union School of Engineering. Expanded about 20%, this new edition includes a new theorem on the reversion of power series; revision of the discussion of permutations, combinations, and elementary probability theory; and a new section on Laplace transforms. Problems relate to the four main fields of engineering. Definitions are given in a thorough and clearcut manner. 1955. 548 pages. $6.50.

THE ELEMENTS OF PROBABILITY THEORY AND SOME OF ITS APPLICATIONS

By Harald Cramér, President of the University of Stockholm. A readable, concise introduction to probability which emphasizes random variables and probability distributions. It also provides a fruitful survey of the important applications—particularly as they affect modern statistical theory and practice. The book calls for a minimum of background knowledge and training. One of the *Wiley Publications in Statistics*, Walter A. Shewhart and Samuel S. Wilks, Editors. 1955. 281 pages. $7.00.

PSYCHOLOGICAL STATISTICS, Second Edition

By Quinn McNemar, Stanford University. Offers an up-to-date discussion of the statistical techniques most frequently used in psychological research. Enlarged for complete coverage, it stresses assumptions and interpretations rather than routine computational procedures. New material in the second edition includes: expansion of the elementary treatment of statistical inference; development of the principles of hypothesis testing by use of the binomial distribution and its normal curve approximation. One of the *Wiley Publications in Psychology*, Herbert S. Langfeld, Advisory Editor. 1955. 408 pages. $6.00.

MATHEMATICS FOR TECHNICAL AND VOCATIONAL SCHOOLS
Fourth Edition

By Samuel Slade and Louis Margolis. The new edition of this widely used text features: (1) an expanded chapter on logarithms and graphs, (2) a rewritten section of differential indexing, (3) many new and revised problems, and (4) modernized format. 1955. 574 pages. $4.48.

Send for on-approval copies.

JOHN WILEY & SONS Inc., 440 Fourth Avenue, New York 16, N.Y.

Two important ® *new texts, coming*

in

APRIL

Neal H. McCoy & R. E. Johnson • **ANALYTIC GEOMETRY**

To insure thorough understanding of basic principles and to give the student an appreciation of the logical structure of mathematical proof the authors of this introductory text have made the proofs of the main theorems especially clear. Illustrative examples are worked out in detail. Principal definitions are set off and numbered for easy reference. There are many graded exercises, with answers to the odd-numbered problems at the back of the book, and useful reference formulas and tables.

in

MARCH

Burton W. Jones • • • **THEORY OF NUMBERS**

Emphasizing *discovery* and the place of mathematical proof in the development of theory, this interesting treatment of the foundations of the number system and the theory of numbers includes problems that show the student how to make discoveries for himself. The use of the Fibonacci sequence to introduce continued fractions is a valuable feature.

Outstanding ® *Recent Texts*

Britton & Snively

ALGEBRA FOR COLLEGE STUDENTS

Revised Edition. 537 pp. $4.50

•

Beaumont & Ball

INTRODUCTION TO MODERN ALGEBRA AND MATRIX THEORY

331 pp. $6.00

Johnson, McCoy & O'Neill

FUNDAMENTALS OF COLLEGE MATHEMATICS

479 pp. $6.00

•

Eves

AN INTRODUCTION TO THE HISTORY OF MATHEMATICS

422 pp. $6.00

from the Publishers of

The Most Legible and Complete Tables Available

RINEHART MATHEMATICAL TABLES, FORMULAS, AND CURVES.

Compiled by Harold Larsen. 280 pp. $2.50

Rinehart & Company, Inc. ® 232 MADISON AVENUE, NEW YORK 16

IBM News FOR MATHEMATICIANS

NEW CAREER OPPORTUNITIES OPEN FOR MATHEMATICIANS

IBM's director of Applied Science, Dr. C. C. Hurd, and Wm. McDowell, V. P. in charge of engineering, discuss a series of calculations to be sped to solution through one of IBM's great electronic data processing machines.

IBM is looking for a special kind of mathematician and will pay especially well for his abilities.

This man is a pioneer, an educator—with a major or graduate degree in Mathematics, Physics, or Engineering with Applied Mathematics equivalent.

Desirable, but not required, is experience in teaching Applied Mathematics and the use of automatic computing equipment.

If you can qualify, you'll work as a special representative of IBM's Applied Science Division as a top-level consultant to business executives and scientists. It is an exciting position, crammed with interest, responsibility.

Employment assignment can probably be made in almost any major U. S. city you choose. Excellent working conditions and employee benefit program.

Your reply will, of course, be held in the strictest confidence. WRITE, giving full details of education and experience, to:

Dr. C. C. Hurd, Director
Dept. 1404
Applied Science Division

INTERNATIONAL BUSINESS MACHINES CORP.
590 Madison Avenue, New York 22, N. Y.

With 20,000 positions for storing data on its magnetic drum, this IBM Type 650 data processing machine is now coming off IBM's production lines. Dozens of these machines will be installed in 1955 for scientific and engineering computation.

IBM
TRADE MARK

Timely McGRAW-HILL Books

DIFFERENTIAL AND INTEGRAL CALCULUS

New Second Edition

By HAROLD M. BACON, Stanford University. 559 pages, $6.00

This is a revision of a highly readable and precise treatment of the theory of calculus. All chapters which served only as a review and contained no calculus are eliminated. The author has rewritten the introduction of limits and the chapter on the differential as well as various other sections to insure the greatest possible clarity and to bring the material up to date. There is a large number of both routine and challenging exercises and problems, many of them new, which help to make the theory meaningful.

ANALYTIC GEOMETRY

New Second Edition

By ROSS R. MIDDLEMISS, Washington University. 326 pages, $3.75

New problems, new figures, new illustrative examples, and the introduction of a new point of view in certain areas continues to develop the students' general mathematical training as well as presenting the basic methods and formulas of the subject. Following a brief introductory chapter reviewing the basic facts and formulas of algebra and trigonometry are 13 chapters on plane analytic geometry and three chapters on solid analytic geometry.

PLANE TRIGONOMETRY

By C. R. WYLIE, JR., University of Utah. 392 pages, $4.00

A sound text which focuses primarily on the analytical aspects of plane trigonometry indicating its importance in a subsequent study of mathematics, physics, or engineering. The style is informal and direct, and the numerous illustrative examples well chosen and completely worked out. In addition to all the usual topics, discussions of complex numbers, trigonometric series, and hyperbolic functions are included.

PRINCIPLES OF MATHEMATICS

By C. B. ALLENDOERFER, University of Washington, and C. O. OAKLEY, Haverford College. Ready in May.

The book is designed for students who have completed a course in intermediate algebra and prepares them for a standard course in calculus. The authors believe that large parts of the standard freshman courses in mathematics are obsolete, and have written this text in the spirit of modern mathematics, including material on logic, the number system, groups, fields, sets, Boolean Algebra, and statistics. Standard material is abbreviated and the computational side is emphasized.

Send for copies on approval

McGraw-Hill Book Company, Inc.

330 West 42nd Street New York 36, N.Y.

The topics of college algebra presented in a clear logical manner

COLLEGE ALGEBRA

Revised Edition

by Paul R. Rider

Retaining the same clear style of presentation which characterized the original edition, this text presents the topics of college algebra and develops an appreciation of mathematical rigor in the student. Due to a simplified introduction and a fuller discussion of fundamental ideas, the student is led more gradually and more easily to the subject. Completely new sets of exercises have been provided and improvements have been made in the method of presenting the subject of generalized exponents. **Published in March**

A unified treatment of vector and tensor analysis

VECTOR and TENSOR ANALYSIS

by Nathaniel Coburn

The subject of vector analysis is treated in the conventional Gibbs manner—that is, by use of directed line segments. However, in order to prepare the student for the study of more general quantities a distributive star product has been introduced in the vector analysis section. Tensors are initiated by studying the direct product of two vectors. The complexity of the direct notation leads naturally to a study of the index notation. Most of the basic ideas are developed for Euclidean three-space; however, in many cases, the corresponding problem for n-dimensional Riemannian space is indicated. **Published in the Spring**

The Macmillan Company

60 FIFTH AVENUE, NEW YORK 11, N.Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN